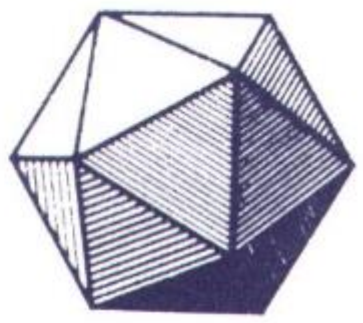
Kaleidoscopic Constructions by Rotational Generators



MAA NCS - Spring 2026 talk, March 28, 2026
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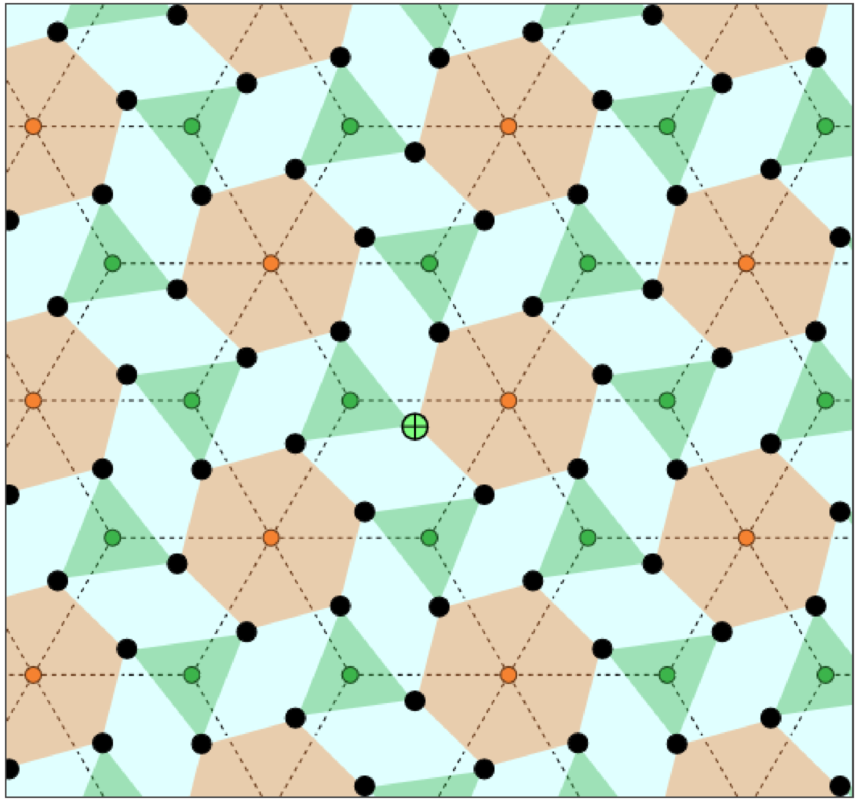
Abstract

Kaleidoscopic constructions of uniform polytopes and tilings with Coxeter groups rely on reflections. We generalize mirrors to discrete rotations to build geometric configurations.

We start with rank-2 signatures ${}_{\langle k\rangle}(g)_d$ being a k-uniform, d-regular symmetric hypergraph, where $g = vd = ke$ is vertex/edge incidence handshake. The kaleidoscopic seed also yields dual ${}_k(g)_{\langle d\rangle}$ (activating d-rotation), and free truncations ${}_{\langle k\rangle}(g)_{\langle d\rangle}$ (both active). The dual of truncation defines bipartite Levi graph.

Principles are illustrated in online interactive javascript, highlighting ordinary dihedral mirror symmetry of isogonal polygons (and isotoxal duals), complex apeirogonal tilings, and rotational generator ${}_p(g)_q$ within 3D chiral polyhedral symmetries $[p,q]+$, order g.



Reflectional Kaleidoscopes

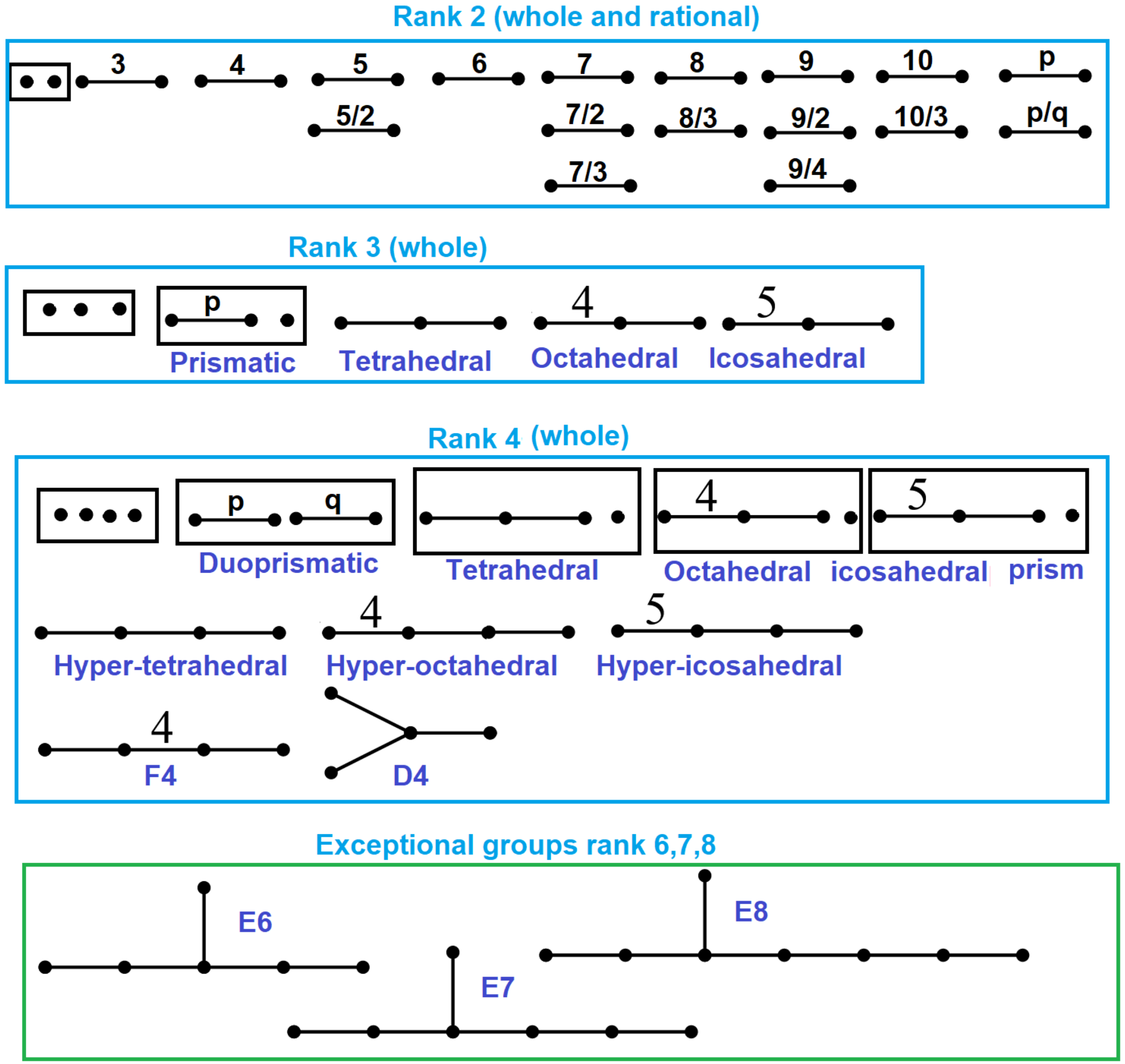
Regular and uniform polytopes can be constructed within families based on ringed permutations on Coxeter diagrams, a seen as a collection of nodes and branches.

Nodes representing *n* mirror hyperplanes (facets) of an (*n*-1)-simplex, and branch order *p* represents the dihedral angle, 180°/*p*, between the hyperplanes (ridges).

- Order 2 (orthogonal) branches are suppressed.
- Order 3 branches are unlabeled.
- Rational orders are allowed *p*/*q*
- *Disconnected graphs produce "prisms"*

Coxeter groups can be finite, Euclidean or hyperbolic.

Finite Coxeter groups are shown (right)



Rank-2 Reflectional Kaleidoscopes – Sector Domains

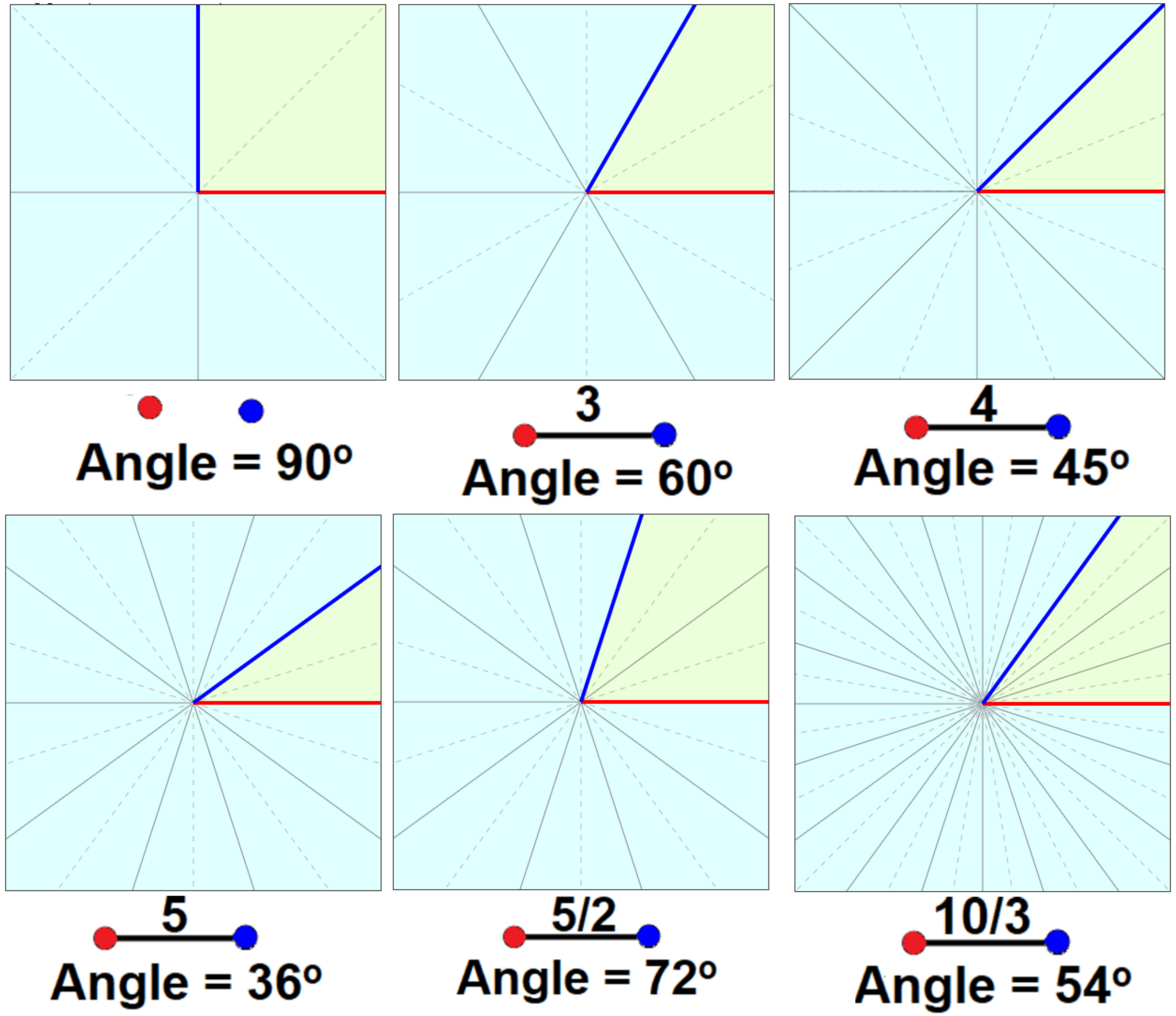
Rank-2 Coxeter groups can be written in:

- Coxeter diagram: O---*p*---O
- Coxeter notation as [*p*] with branch order
- Signature notation ${}_2(2p)_2$ with explicit mirror orders.
- Its structure is dihedral group, Dih(*p*) order 2*p.*

Wythoff construction (isogonal polygons):

- Each node is ringed (active) or unringed (inactive).
- Ringed node permutations:

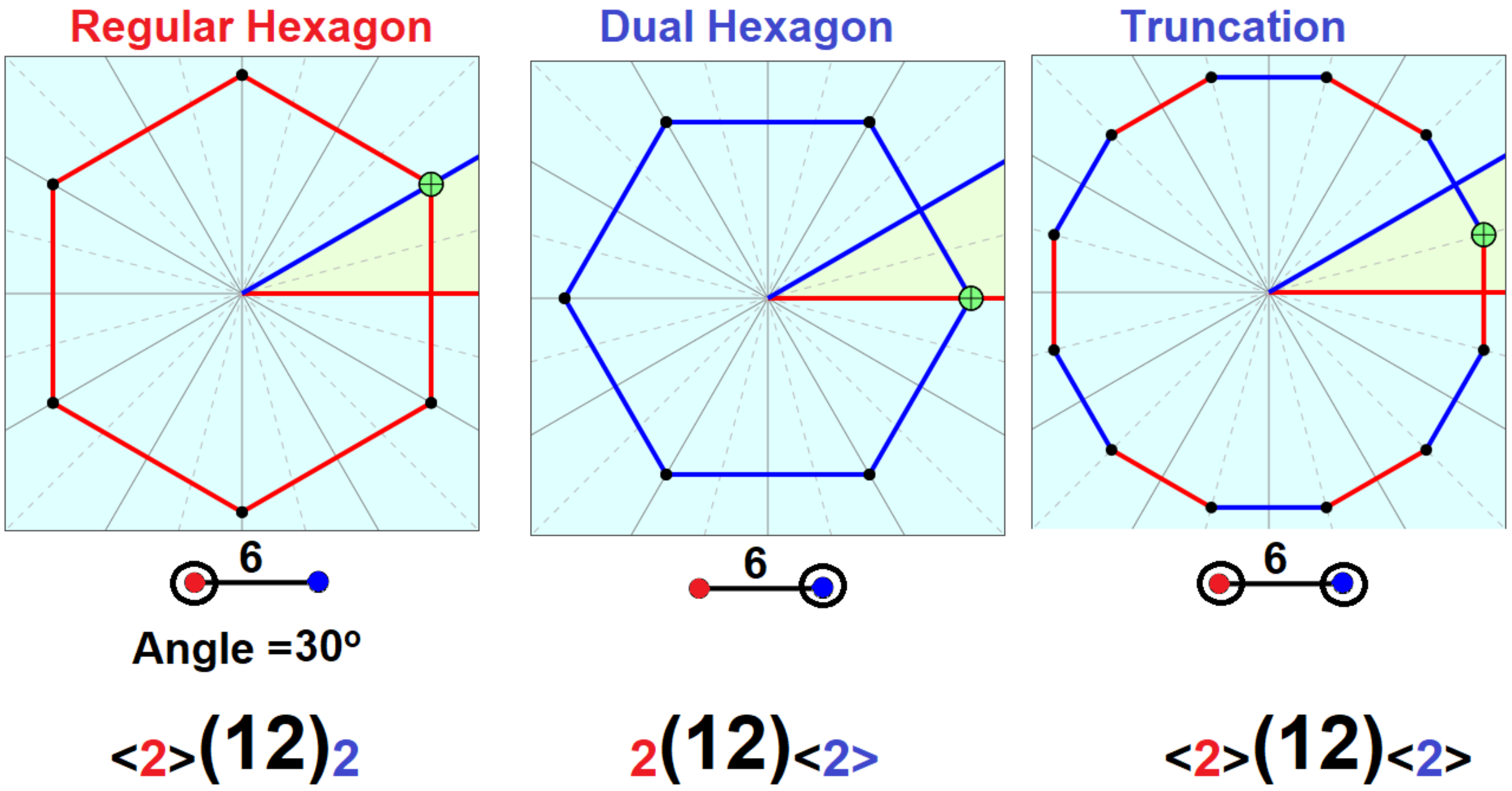
1. ${}_{\langle 2\rangle}(2p)_2$ – first mirror active
2. ${}_2(2p)_{\langle 2\rangle}$ – second active (dual)
3. ${}_{\langle 2\rangle}(2p)_{\langle 2\rangle}$ – both active (truncation)



Rank 2 Kaleidoscope: Isogonal Polygons

A Kaleidoscope uses one “seed” point that is reflected across the mirrors.

Coloring mirrors red and blue show a red hexagon, its dual blue hexagon, and a uniform truncation (alternation both red and blue edges.)

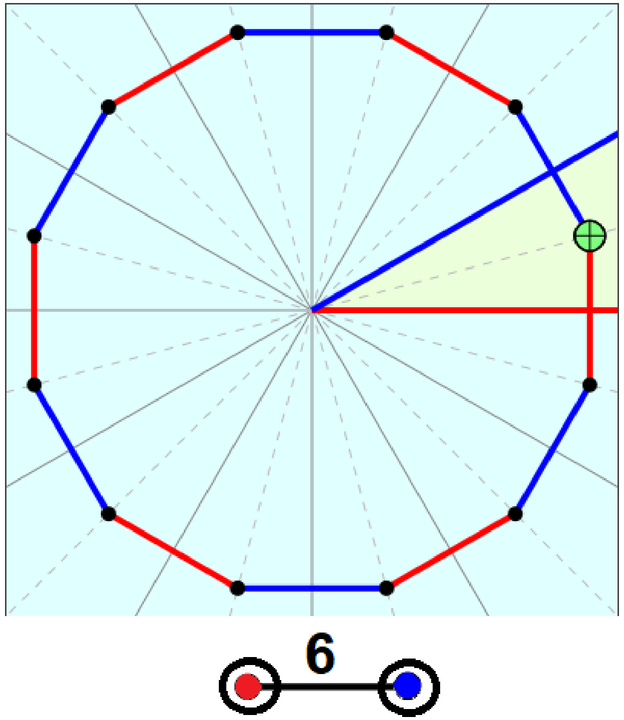


Symmetry Doubling – Fundamental Domain Bisected

If the generator point is exactly between the red and blue mirror (uniform truncation), an inactive (green) mirror can be added on the bisection which maps the red and blue mirrors together.

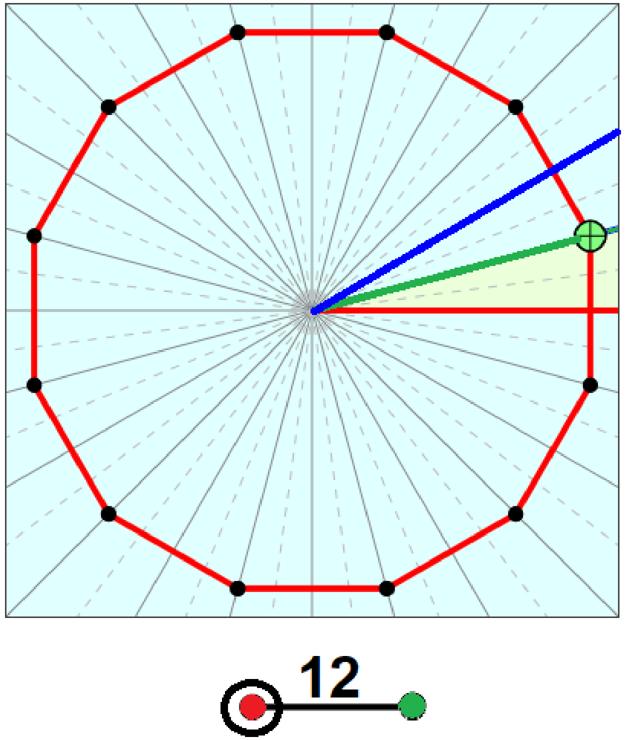
$${}_{\langle 2\rangle}(2p)_{\langle 2\rangle} = {}_{\langle 2\rangle}(4p)_{2}$$

Uniform Truncation



$${}_{\langle 2\rangle}(12)_{\langle 2\rangle}$$

Doubling by middle

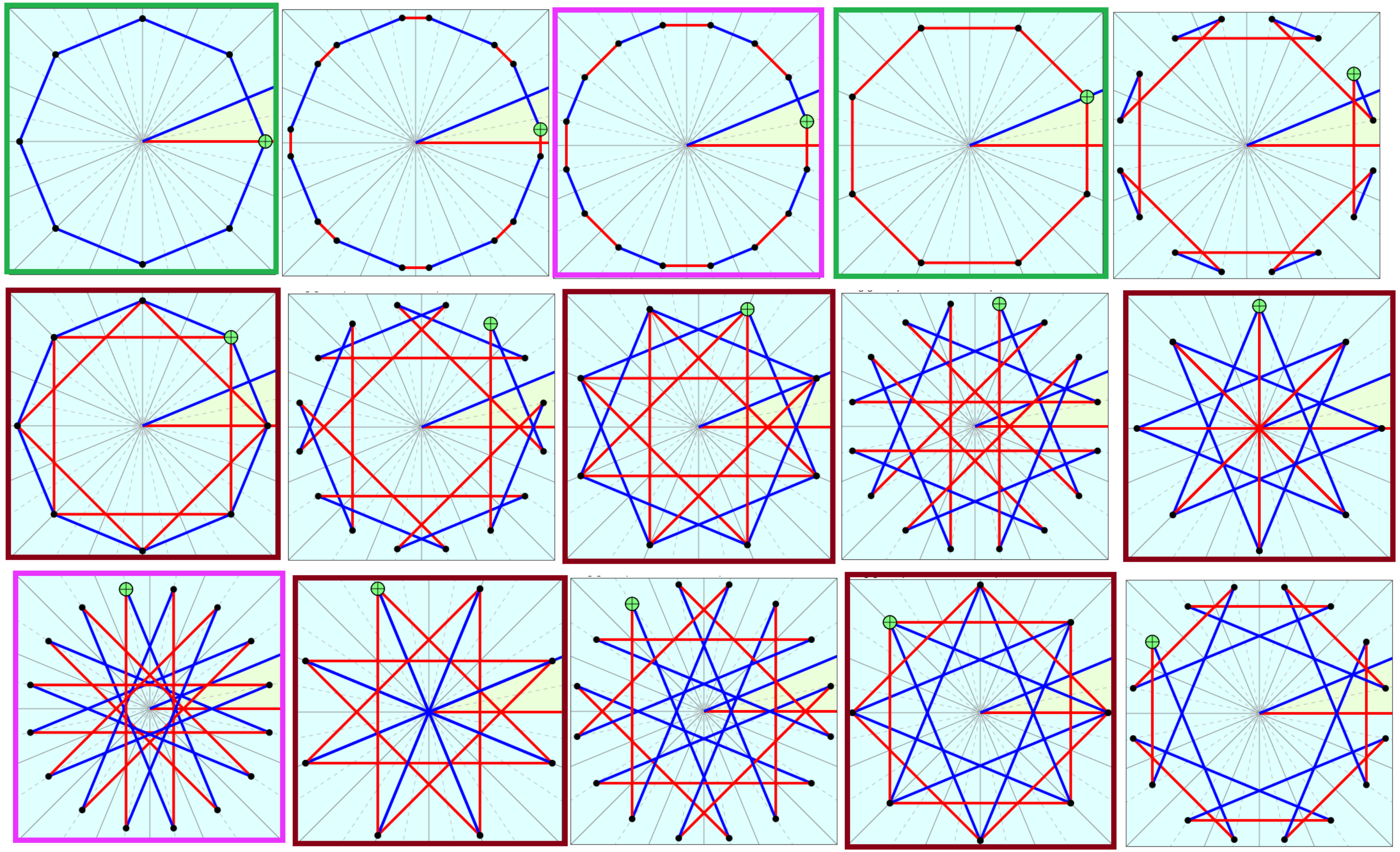


$${}_{\langle 2\rangle}(24)_{2}$$

"Anti-truncations" – Seed Outside Fundamental Sector

Examples: [8], $_{<2>}(16)_{<2>}$ Truncated octagons: 0 | 1/2 | 1 | 3/2 | 2 | 5/2 | 3 | 7/2 | 4 | 9/2

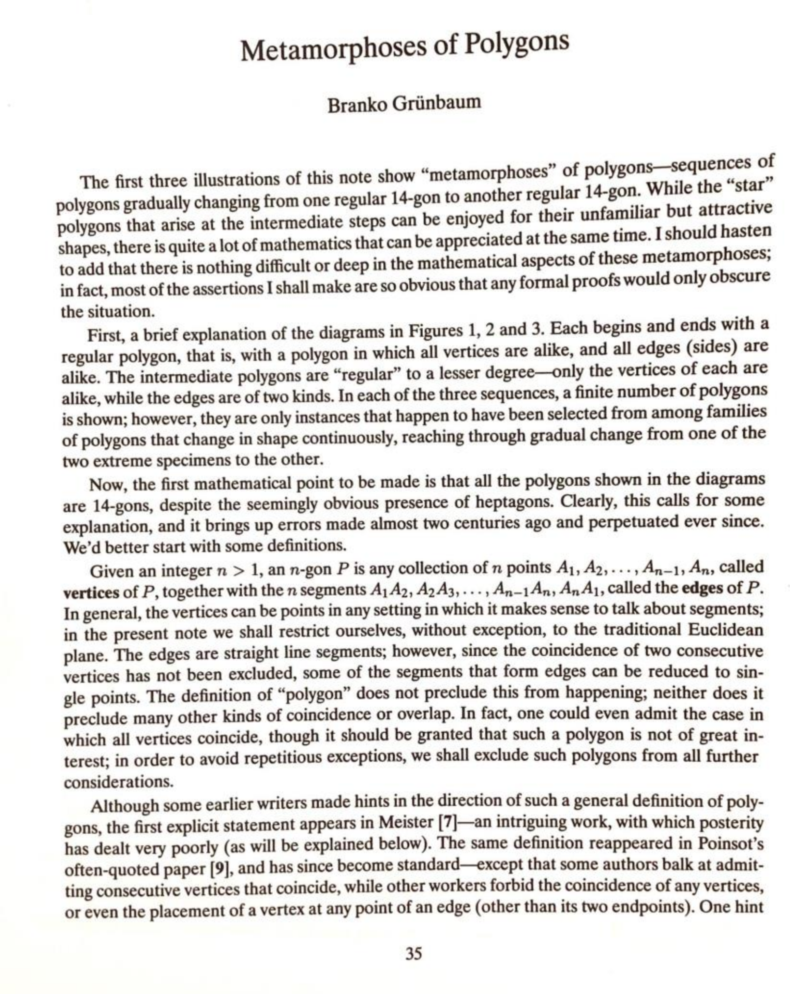
Note: *Vertices may coincide when seed is positioned on a virtual mirror.*



Isogonal / Isotoxal Starry Polygons t'{*p*/*q*}

Branko Grünbaum, *Metamorphoses of Polygons*, *The Lighter Side of Mathematics*, 1994

Isogonal (vertex-transitive, 2 edge types) polygons and i**sotoxal** (duals, edge-transitive, 2 vertex types)



Metamorphoses of Polygons

Branko Grünbaum

The first three illustrations of this note show "metamorphoses" of polygons—sequences of polygons gradually changing from one regular 14-gon to another regular 14-gon. While the "star" polygons that arise at the intermediate steps can be enjoyed for their unfamiliar but attractive shapes, there is quite a lot of mathematics that can be appreciated at the same time. I should hasten to add that there is nothing difficult or deep in the mathematical aspects of these metamorphoses; in fact, most of the assertions I shall make are so obvious that any formal proofs would only obscure the situation.

First, a brief explanation of the diagrams in Figures 1, 2 and 3. Each begins and ends with a regular polygon, that is, with a polygon in which all vertices are alike, and all edges (sides) are alike. The intermediate polygons are "regular" to a lesser degree—only the vertices of each are alike, while the edges are of two kinds. In each of the three sequences, a finite number of polygons is shown; however, they are only instances that happen to have been selected from among families of polygons that change in shape continuously, reaching through gradual change from one of the two extreme specimens to the other.

Now, the first mathematical point to be made is that all the polygons shown in the diagrams are 14-gons, despite the seemingly obvious presence of heptagons. Clearly, this calls for some explanation, and it brings up errors made almost two centuries ago and perpetuated ever since. We'd better start with some definitions.

Given an integer $n > 1$, an n-gon P is any collection of n points $A_1, A_2, \ldots, A_{n-1}, A_n$, called **vertices** of P, together with the n segments $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1$, called the **edges** of P. In general, the vertices can be points in any setting in which it makes sense to talk about segments; in the present note we shall restrict ourselves, without exception, to the traditional Euclidean plane. The edges are straight line segments; however, since the coincidence of two consecutive vertices has not been excluded, some of the segments that form edges can be reduced to single points. The definition of "polygon" does not preclude this from happening; neither does it preclude many other kinds of coincidence or overlap. In fact, one could even admit the case in which all vertices coincide, though it should be granted that such a polygon is not of great interest; in order to avoid repetitious exceptions, we shall exclude such polygons from all further considerations.

Although some earlier writers made hints in the direction of such a general definition of polygons, the first explicit statement appears in Meister [7]—an intriguing work, with which posterity has dealt very poorly (as will be explained below). The same definition reappeared in Poinsot's often-quoted paper [9], and has since become standard—except that some authors balk at admitting consecutive vertices that coincide, while other workers forbid the coincidence of any vertices, or even the placement of a vertex at any point of an edge (other than its two endpoints). One hint

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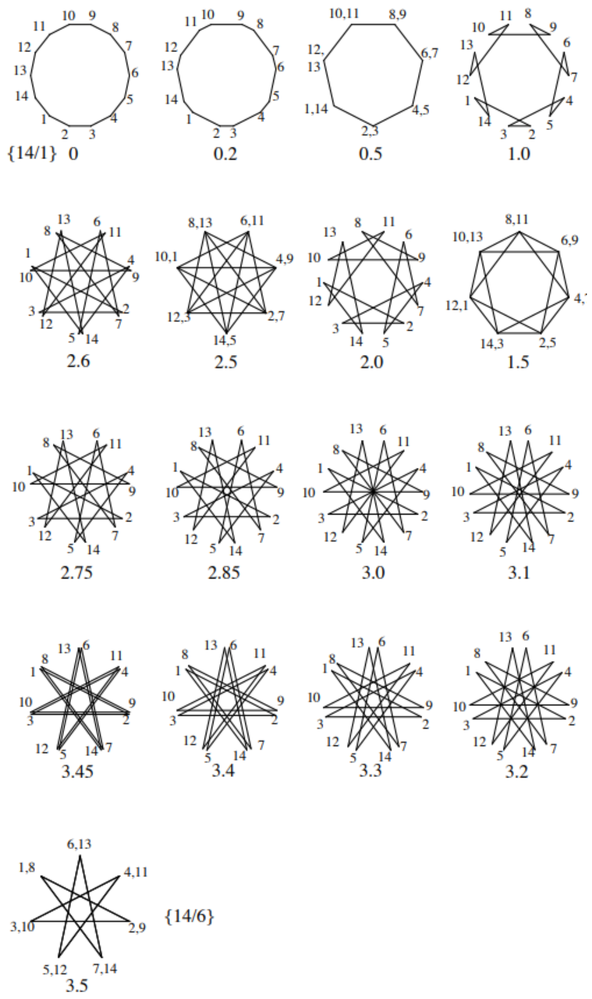
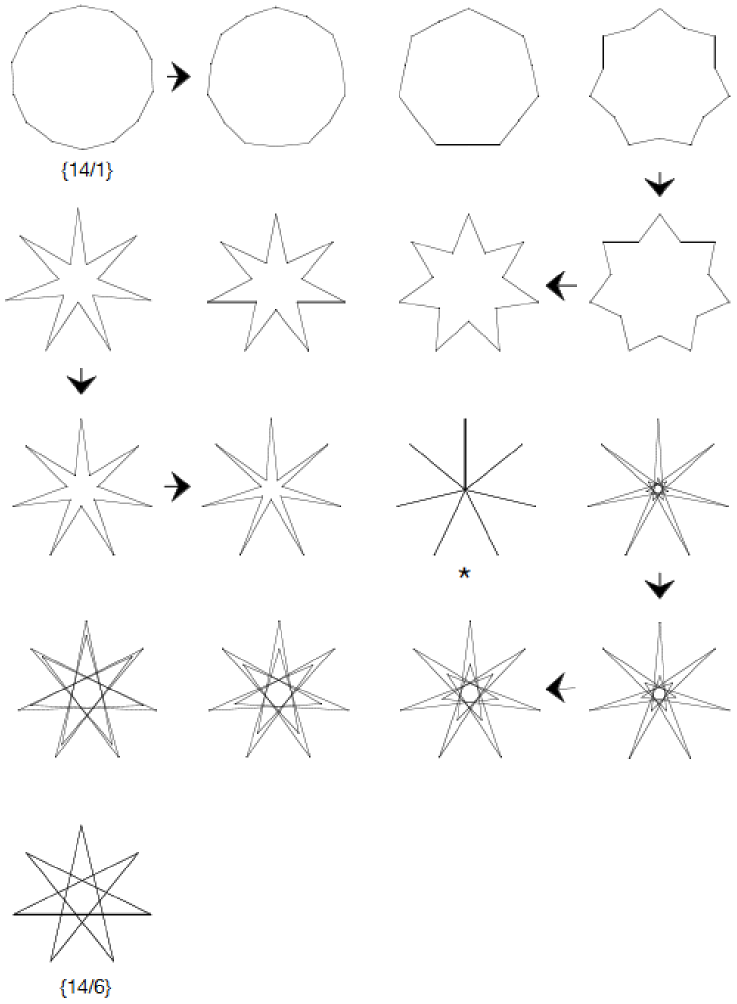




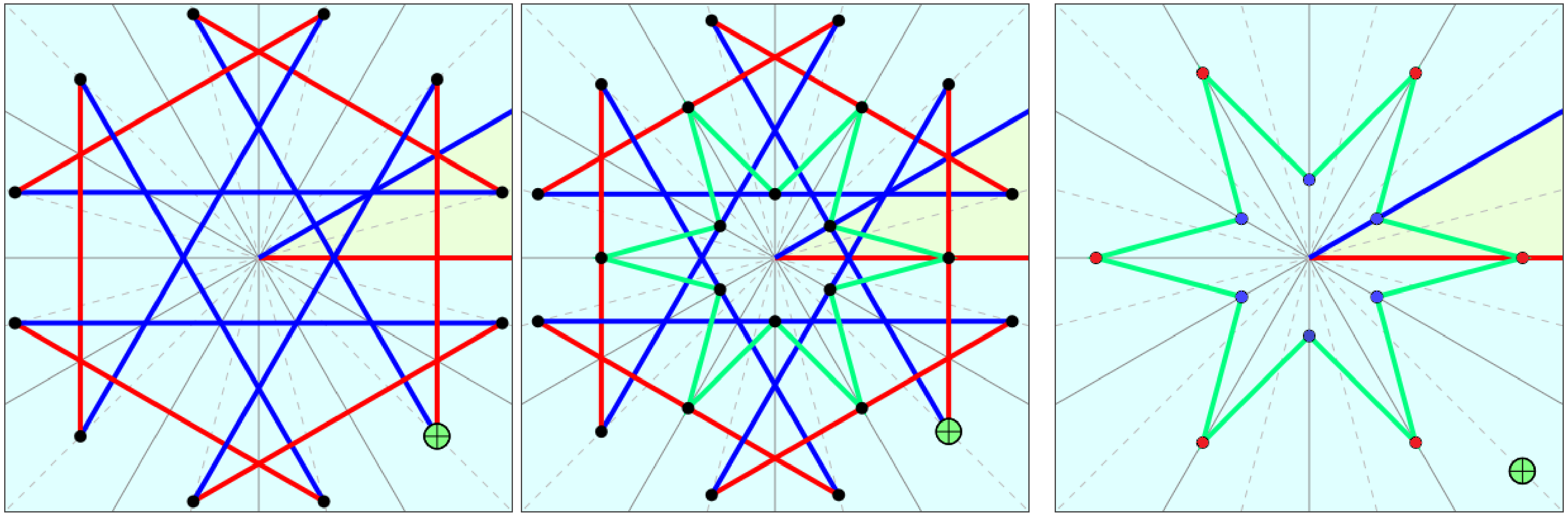
Isotoxal Polygons as Duals of Truncations

(Truncations) Isogonal (vertex-transitive, 2 edge types) polygons, t'{6}

Dual is **Isotoxal** (edge-transitive, 2 vertex types), dt'{6}

Incidence matrices can show the 2 edge or vertex types in a 3x3 matrix.

Antitruncation | **With dual (edge center)** | **Dual**



Isogonal

12	1	1
2	6	-
2	-	6

Isotoxal

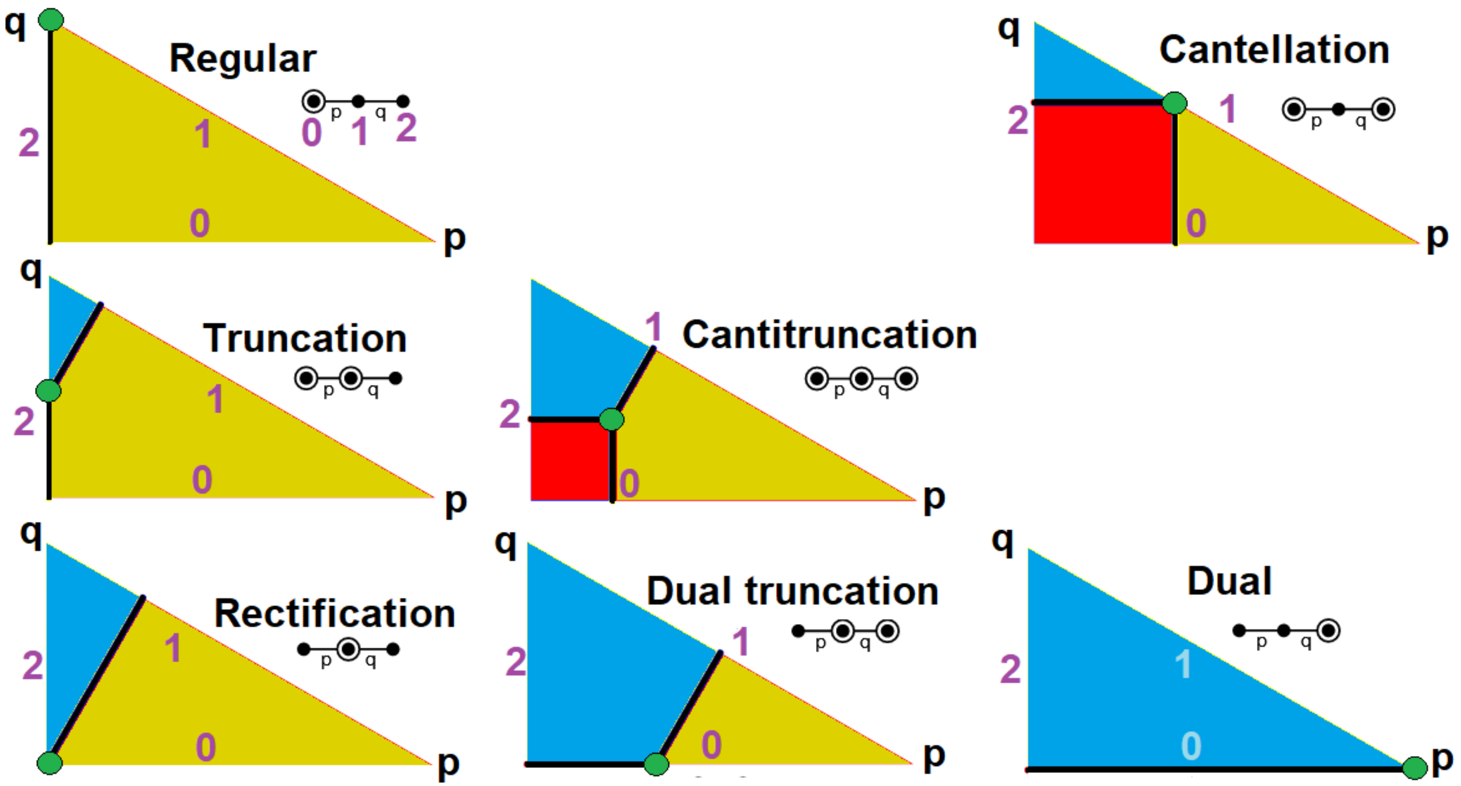
6	-	2
-	6	2
1	1	12

Rank-3 Kaleidoscopic Uniform Constructions

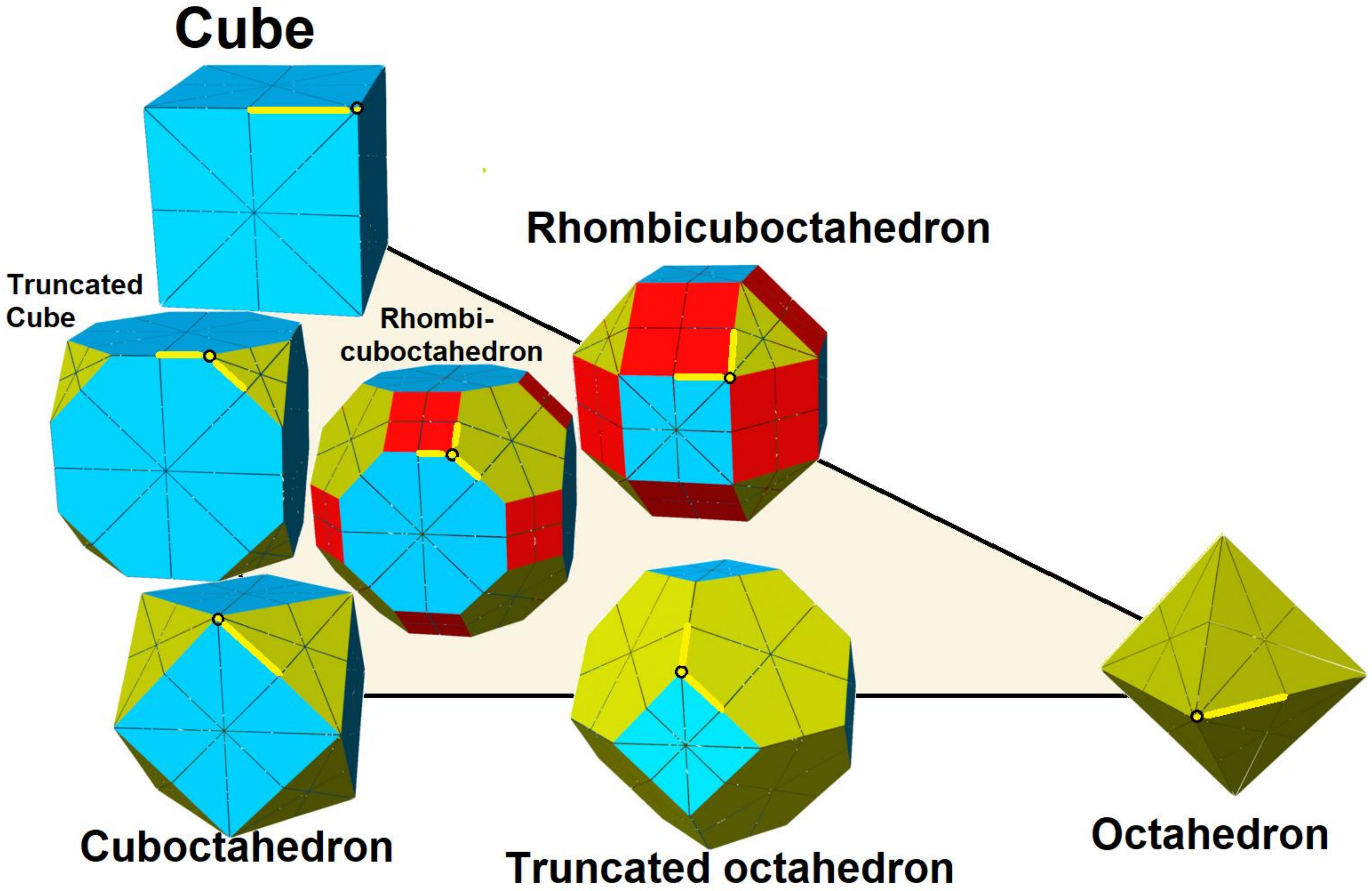
Coxeter diagram: O—*p*—O—*q*—O, bracket notation [*p*,*q*] and incidence notation ${}_2(2p)_2(2q)_2$.

They are constructed by 3 mirror generators corresponding to the 3 nodes, index {0,1,2}.
Finite forms include polyhedral groups [3,3], [3,4], [3,5], $1/p+1/q<1/2$.
Euclidean forms are [4,4], and [3,6], $1/p+1/q=1/2$.
Hyperbolic forms include [4,5], [3,7], [5,5], … [*p*,*q*] with $1/p+1/q>1/2$

There are 7 ring permutations that generate uniform polyhedra or tilings.



Uniform Polyhedra in Cube/Octahedral Family

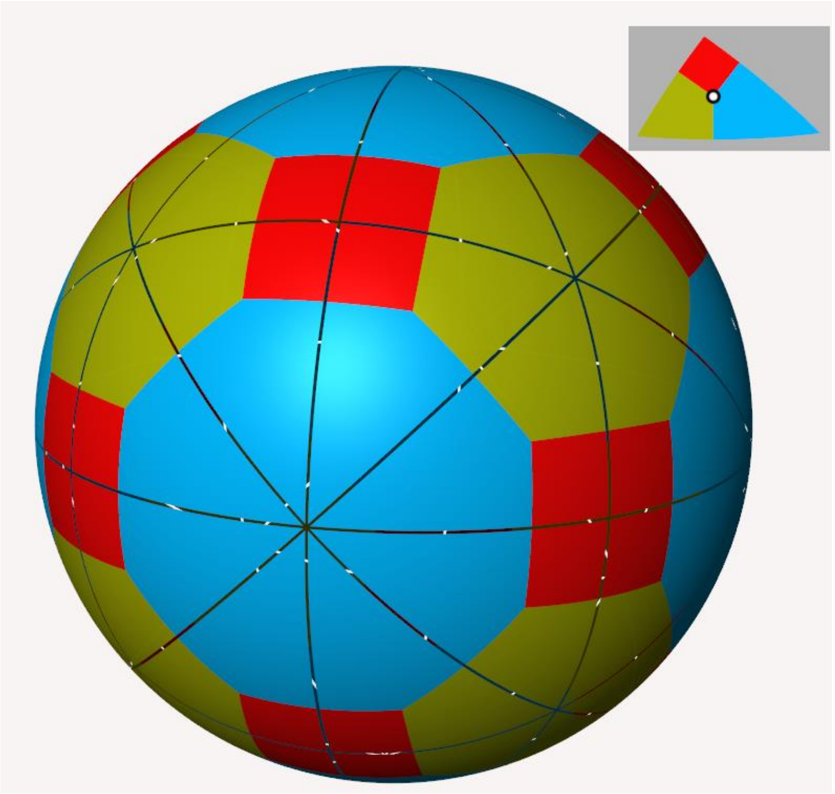


KaleidoTile Software

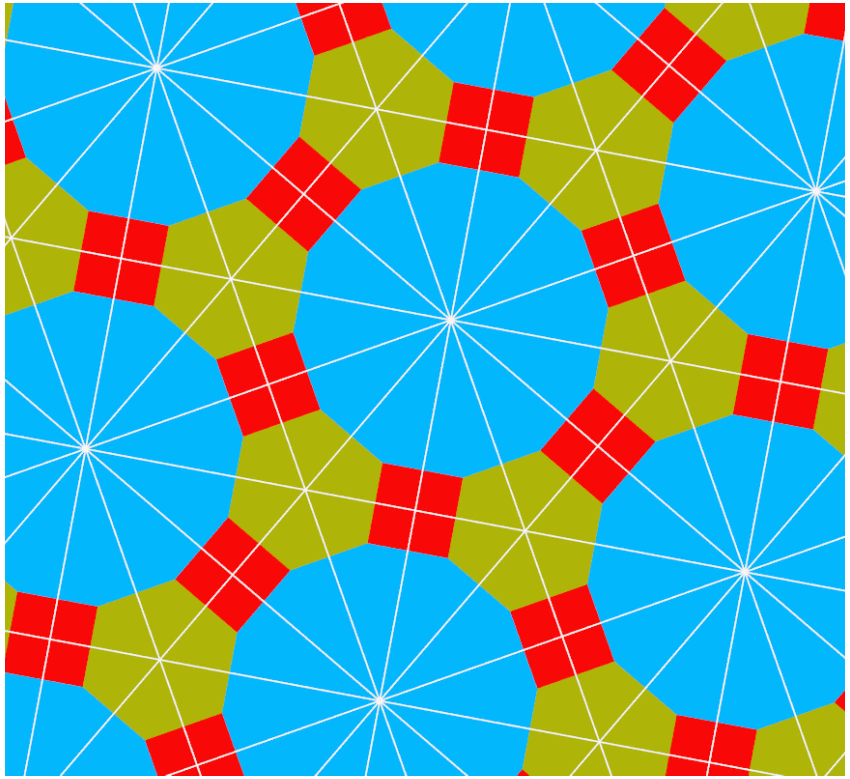
https://www.geometrygames.org/KaleidoTile

Jeff Weeks' free educational software allows any (**p q r**) construction by movable seed point.

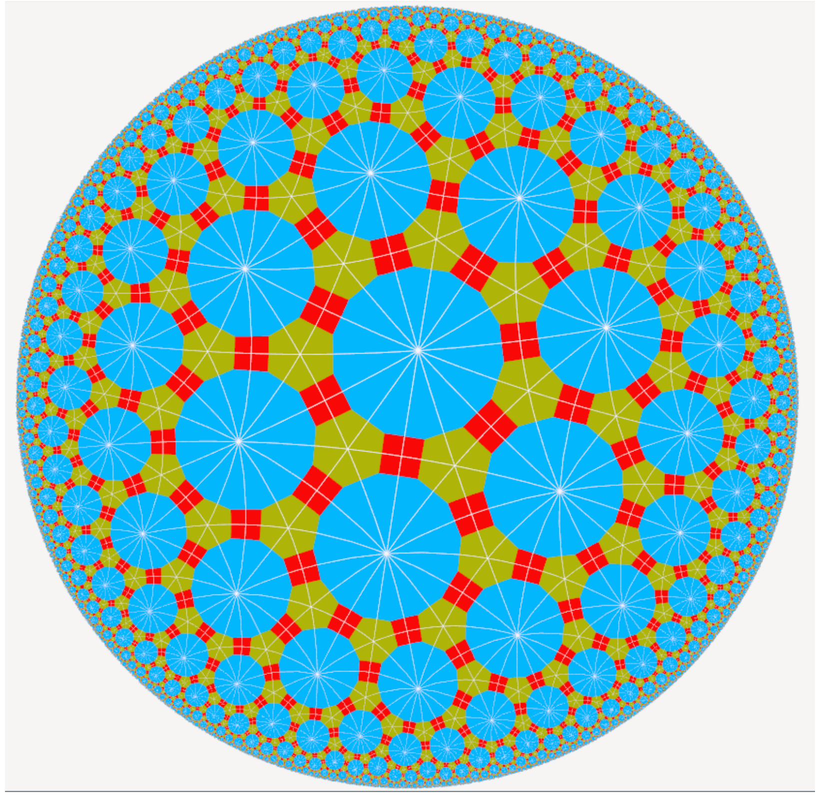
Spherical (4 3 2) (polyhedral)



Euclidean (6 3 2)



Hyperbolic (7 3 2)



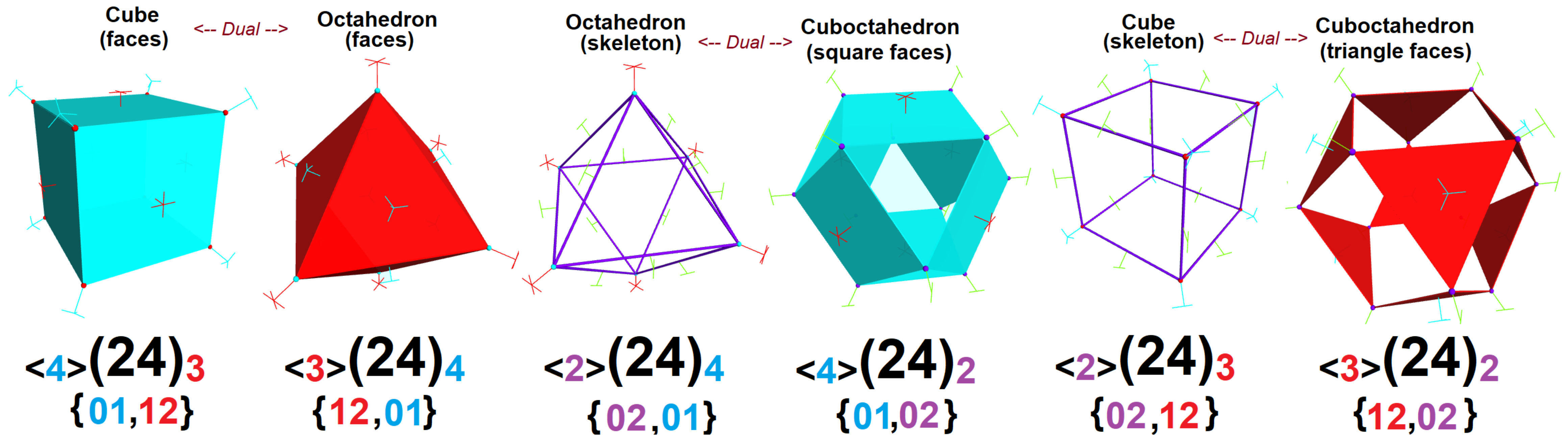
Rotational Generators via 3 Reflections

Rank 3 Coxeter group [*p*,*q*], or $_2(2p)_2(2q)_2$ has 3 mirror generators indexed {0,1,2}.

The rotational form $[p,q]^+$ has 3 rotational generators (*p*,*q*,2) represented by mirror products {01,12,02}. Products of two generators produces unneeded third generator.

This allows us to define THREE rank 2 signatures: $_p(g)_q$, as well as $_p(g)_2$ and $_q(g)_2$.

Octahedral (cubic) $_2(8)_2(6)_2$ has 6 forms: $_{\langle 4\rangle}(24)_3$ + $_{\langle 3\rangle}(24)_4$ | $_{\langle 2\rangle}(24)_4$ + $_{\langle 4\rangle}(24)_2$ | $_{\langle 2\rangle}(24)_3$ + $_{\langle 3\rangle}(24)_2$



What are these rank-2 signatures? ${}_{p}(g)_{r}$

The notation ${}_{p}(g)_{r}$ come from G. C. Shephard who explored Regular complex polygons in 1952, existing in Complex plane C^2 or Real R^4.

Generators are unitary rotations {R,S} with $\mathbf{R}^p = \boldsymbol{S}^r = \mathbf{I}$.

The polygon ${}_{p}(g)_{r}$ has g/p edges, and g/r vertices.

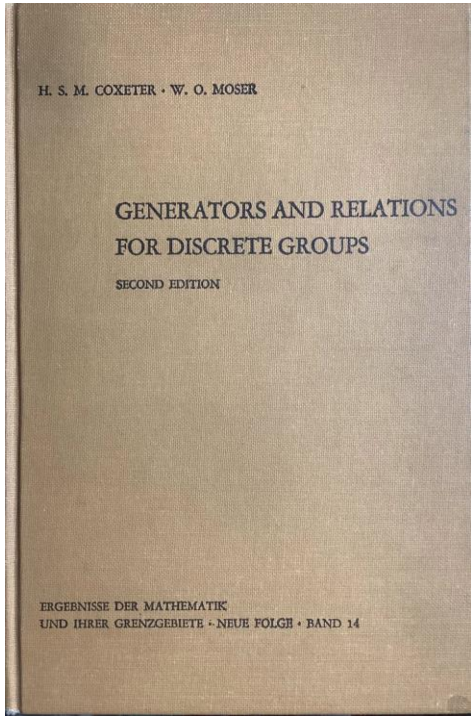
The dual ${}_{r}(g)_{p}$ has g/r edges, and g/p vertices.

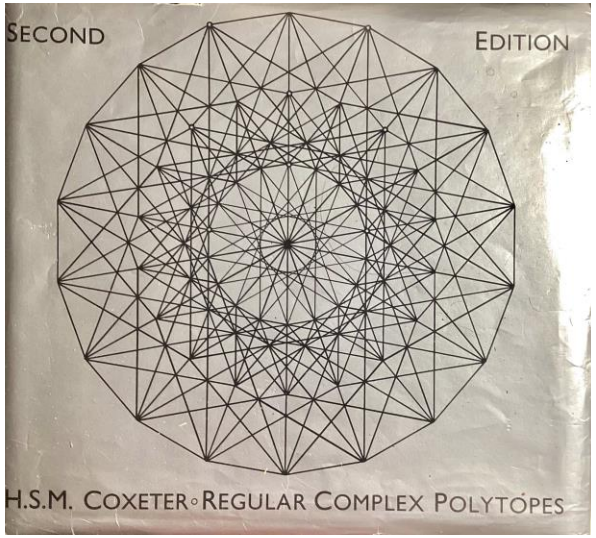
Coxeter notation $p[q]r$, *polygon* $p\{q\}r$, with theory to match with Coxeter group $[q] = {}_{2}[q]_{2}$, and polygon $\{q\} = {}_{2}\{q\}_{2}$.

$g=2h^2/q$, with $2/h = 1/p + 1/r + 2/q - 1$.

See

- Generators and Relation for Discrete Groups, Coxeter and Moser, 1966
- Regular Complex Polygons, Coxeter, 1991



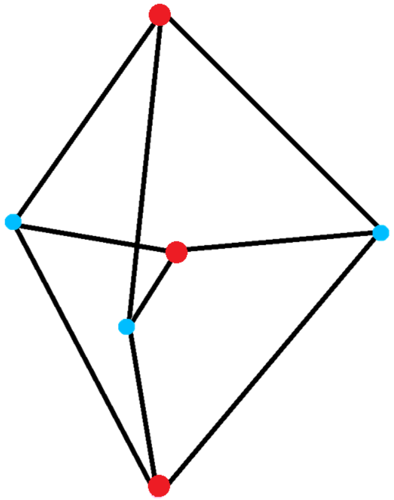


Regular Complex Polygons

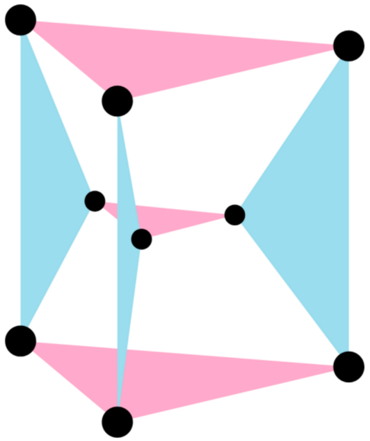
There are 22 forms of regular complex polygons, 8 dual pairs and 6 self-dual (1 real):

${}_m\{4\}_2$ + ${}_2\{4\}_m$, ${}_3\{6\}_2$ + ${}_2\{6\}_3$, ${}_3\{8\}_2$ + ${}_2\{8\}_3$, ${}_4\{6\}_2$ + ${}_2\{6\}_4$, ${}_4\{4\}_3$ + ${}_3\{4\}_4$, ${}_3\{10\}_2$ + ${}_2\{10\}_3$, ${}_5\{6\}_2$ + ${}_2\{6\}_5$, ${}_5\{4\}_3$ + ${}_3\{4\}_5$, ${}_3\{4\}_3$, ${}_3\{3\}_3$, ${}_4\{3\}_4$, ${}_5\{3\}_5$, ${}_3\{5\}_3$, and real ${}_2\{m\}_2$.

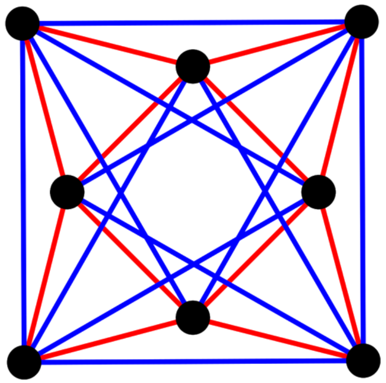
Self-dual forms ${}_p\{q\}_p$ allow odd q, just like real ${}_2\{q\}_2$ *as double covering of generators*.



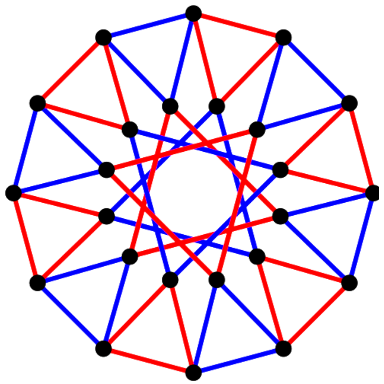
${}_3\{4\}_2 = {}_3\{\} + {}_3\{\}$



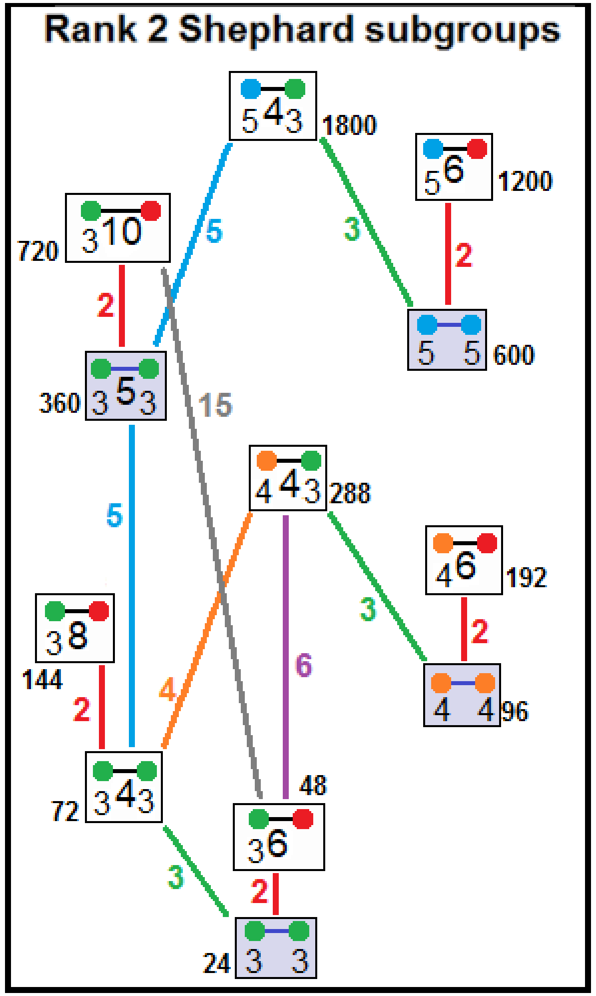
${}_2\{4\}_3 = {}_3\{\} \times {}_3\{\}$



${}_3\{3\}_3$



${}_3\{6\}_2 = {}_{<3>}\{3\}_{<3>}$

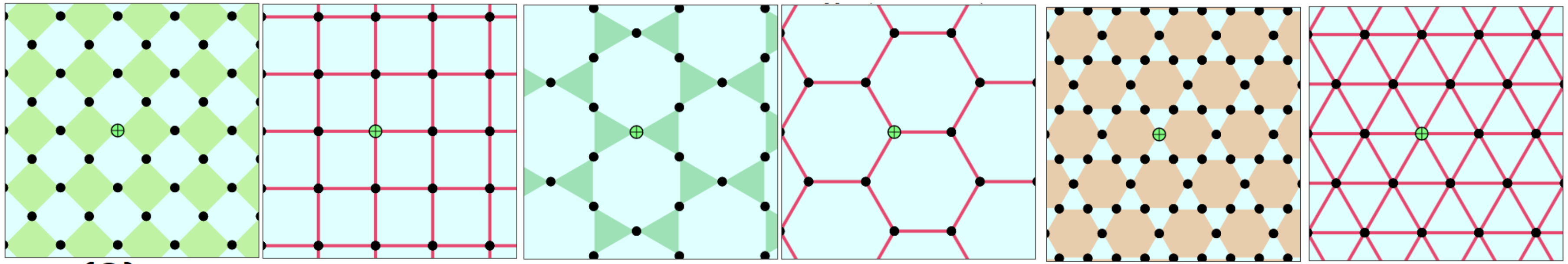


Regular Complex Apeirogons

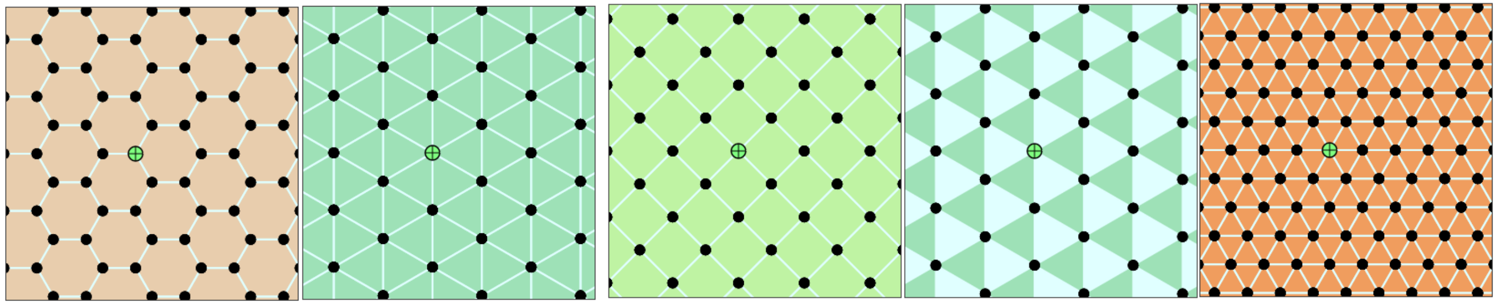
Complex polygons become infinite, spanning C^1 or R^2 with: $1/p + 1/r + 2/q = 1$.

There are 11 forms of regular complex apeirogons in 4 dual pairs and 3 self-duals:
${}_4\{8\}_2 + {}_2\{8\}_4$, ${}_3\{12\}_2 + {}_2\{12\}_3$, ${}_6\{6\}_2 + {}_2\{6\}_6$, ${}_6\{4\}_3 + {}_3\{4\}_6$, ${}_4\{4\}_4$, ${}_3\{6\}_3$, and ${}_6\{3\}_6$.

All can be constructed in R^2 within [4,4] and [3,6], square and tri/hexagonal tilings.



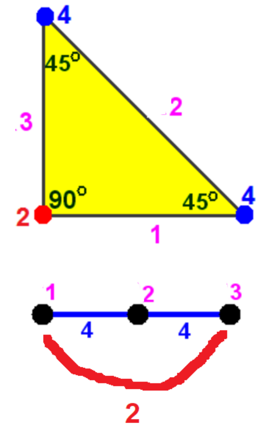
${}_4\{8\}_2$ ${}_2\{8\}_4$ ${}_3\{12\}_2$ ${}_2\{12\}_3$ ${}_6\{6\}_2$ ${}_2\{6\}_2$



${}_6\{4\}_3$ ${}_3\{4\}_6$ ${}_4\{4\}_4$ ${}_3\{6\}_3$ ${}_6\{3\}_6$

Square tiling

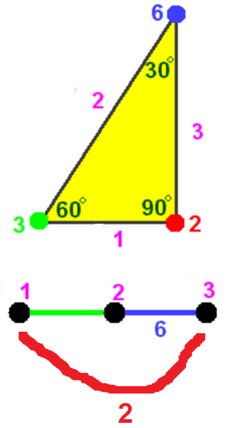
[4,4]



${}_2(8)_2(8)_2$

Triangular tiling

[3,6]



${}_2(6)_2(12)_2$

Varied Presentations of Rotational Edges

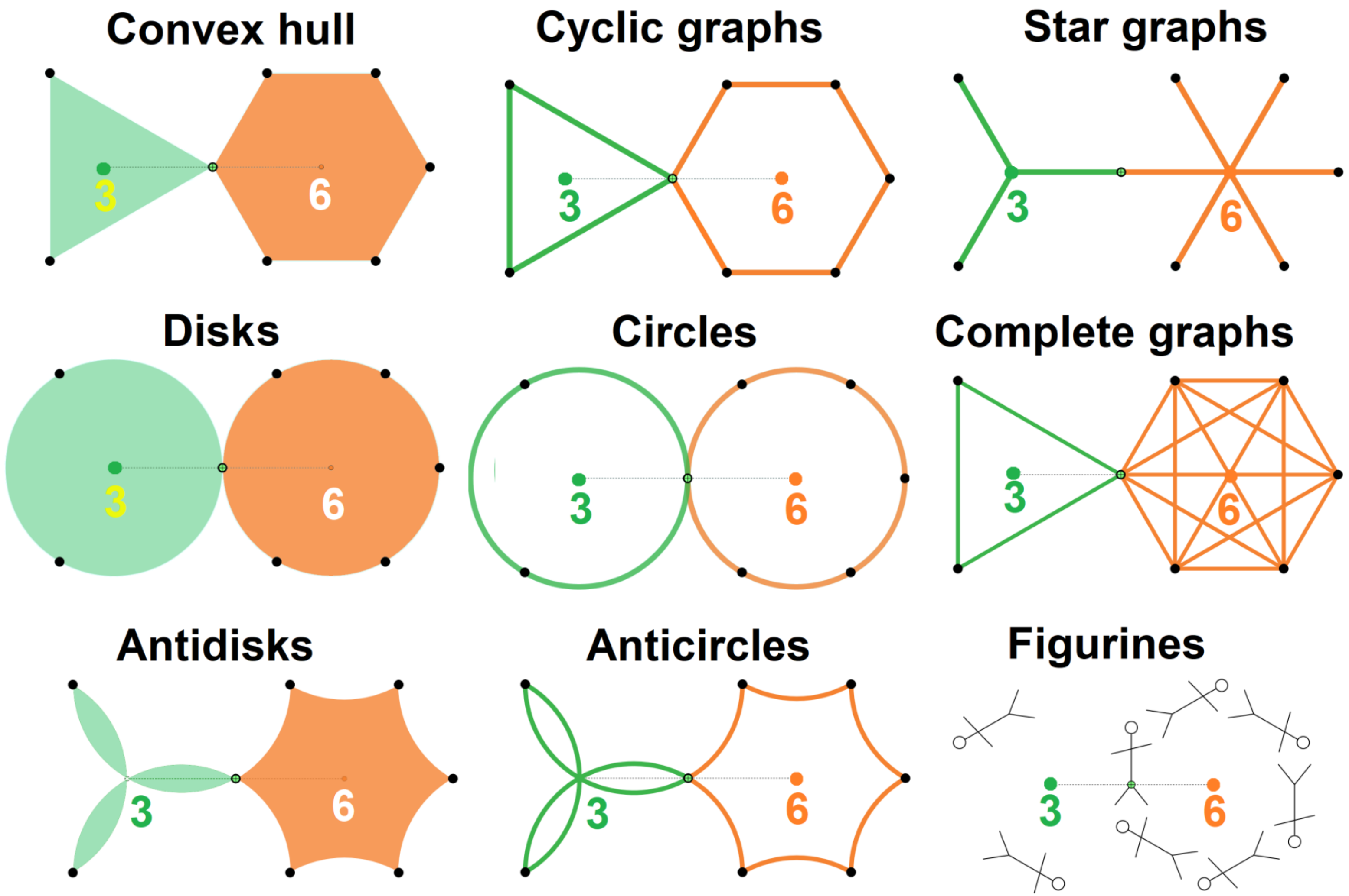
Rotational edges and have no *correct* representation.

Options:

1. Interior area or perimeter
2. Star graphs with center
3. Complete graphs
4. Disks or Circles
5. Anti-disks or Anti-circles

Vertices can be represented by extended “figurines” that rotate and translate with seed.

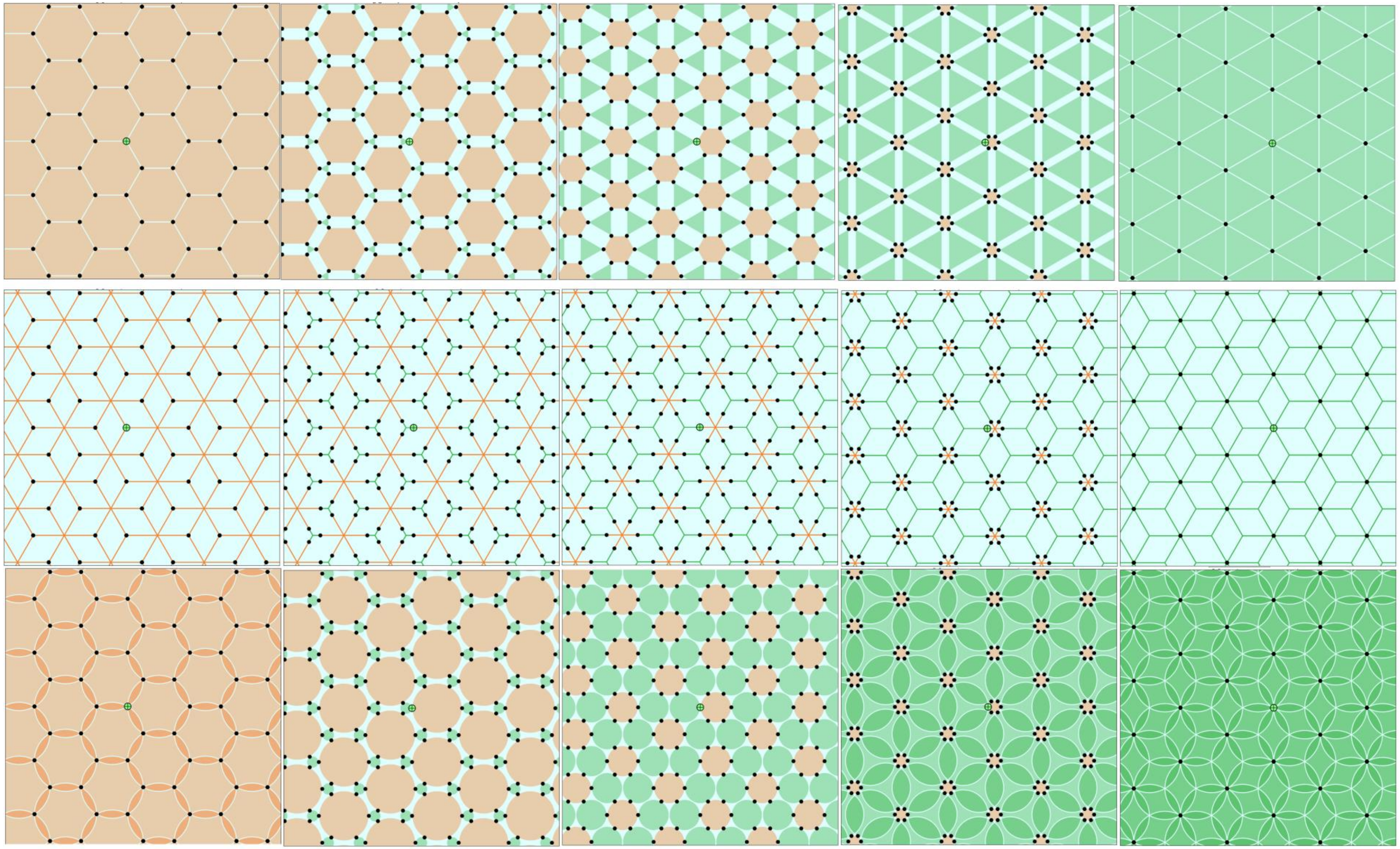
Examples show for ${}_3\{4\}_6$ primary generators.



Truncations of Complex Apeirogons

Truncations in 5 steps from ${}_{<6>}\{4\}_3$ to ${}_{<6>}\{4\}_{<3>}$ to dual ${}_6\{4\}_{<3>}$.

3 rows representations as convex hull, star graphs and disks.

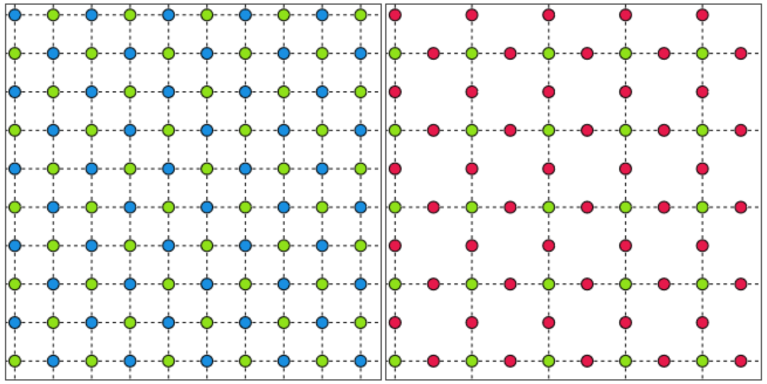


Dual of Truncations are Isotoxal Apeirogons

The dual of an isogonal complex apeirogon is isotoxal (edge-transitive), the Levi graph of regular forms.

Vertices are rotation centers connected by simple edges. Self-dual forms can double ${}_{<p>}\{q\}_{<p>} = {}_{p}\{2q\}_{2}$.

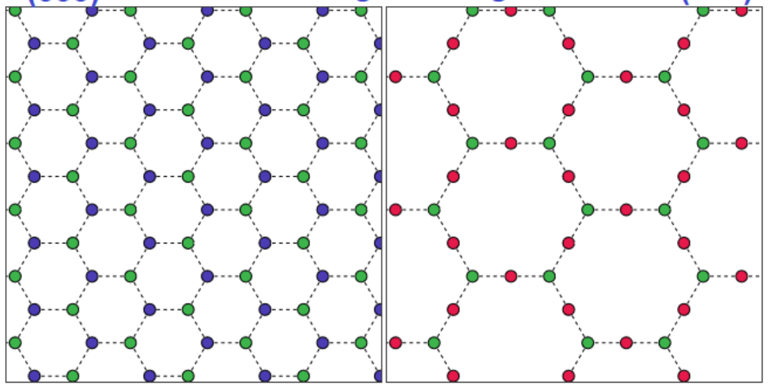
Square tilings (442)



4[4]4 → doubling → 4[8]2

[4,4] {0,1,2} --> {01,12} and {01,1212}

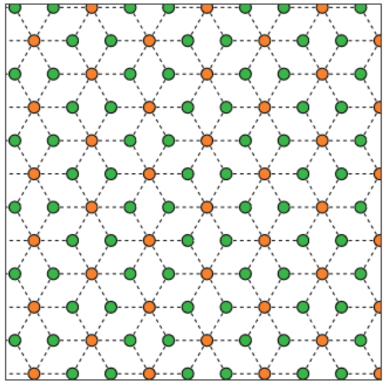
(333) Hexagonal tilings (632)



3[6]3 → doubling → 3[12]2

[6,3] {0,1,2} --> {0101,12} and {0101,02}

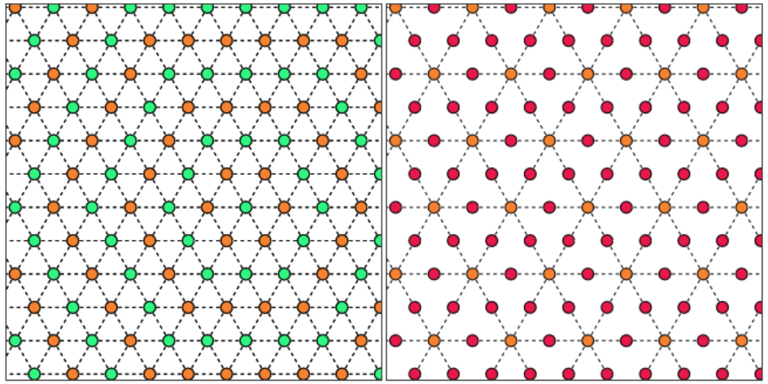
Rhombille (632)



6[4]3

[6,3] {0,1,2} --> {01,12}

(632) Triangular tilings (632)



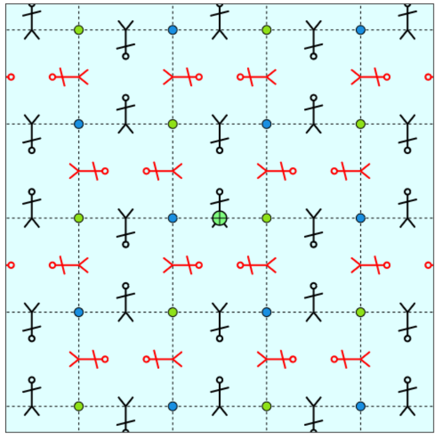
6[3]6 → doubling → 6[6]2

[6,3] {0,1,2} --> {01,2012} and {01,02}

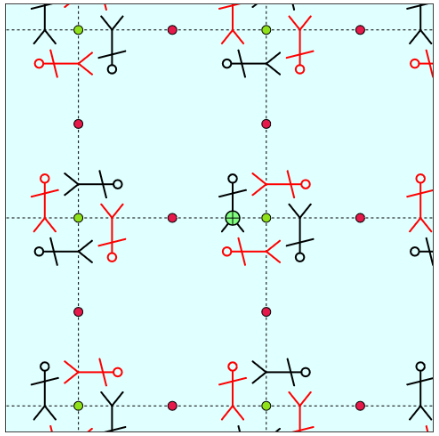
Rotational Stick Figures

We can position stick figures at vertices who rotation with generators.

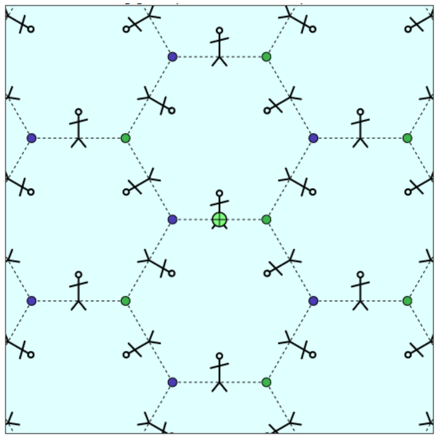
For $_p[q]_r$ cases where p,r both even, we can globally color **red**/**black** by parity.



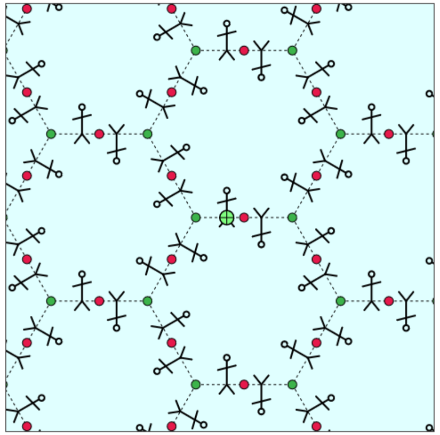
$_4[4]_4$



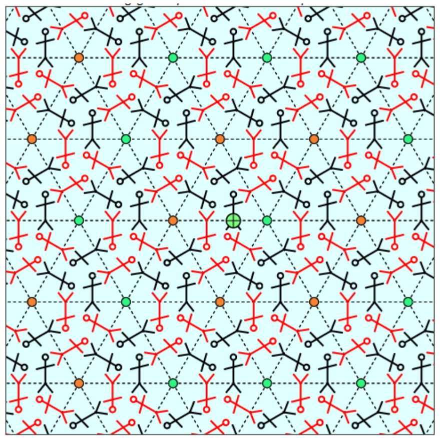
$_4[8]_2$



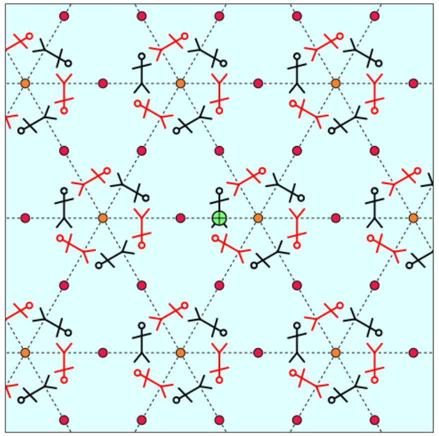
$_3[6]_3$



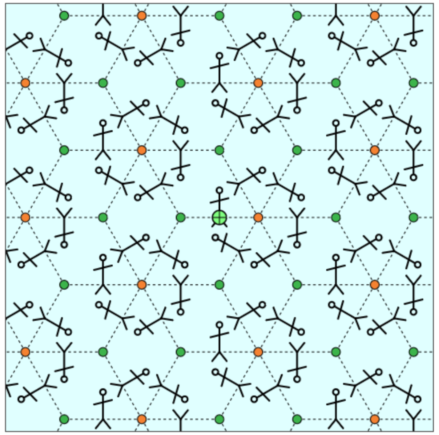
$_3[12]_2$



$_6[3]_6$

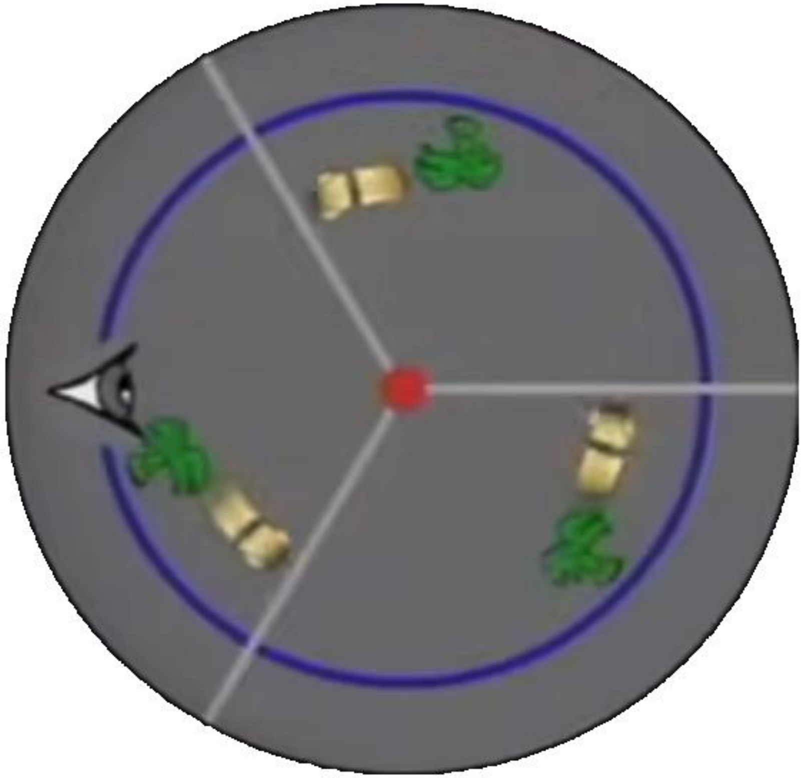


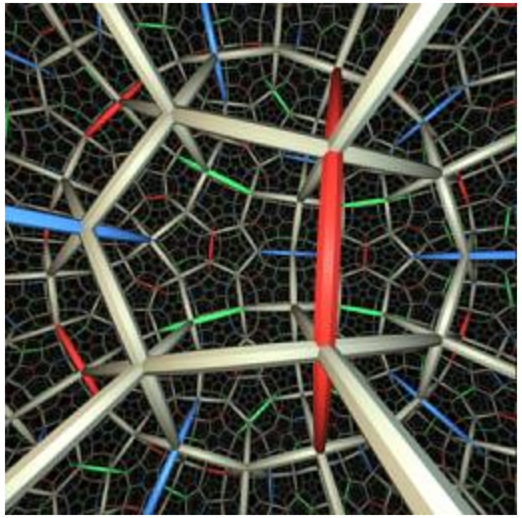
$_6[6]_2$



$_6[4]_3$

Cone Points– sector repeated by rotation





Not Knot, Geometry Center, University of Minnesota, 1991
https://youtu.be/QcLfb0PhfO0?t=338

Rank-3 Rotational Generators (future)

Tesseract incidence:

[**16** 4 6 **4**]
[2 32 3 3]
[4 4 24 2]
[8 12 6 **8**]

[**16** **4**]
[**8** **8**]

Cell-Vertex Signature ${}_{<8>}(64)_{4}$

Facet (cube) and Vertex figure (tetrahedron)

[**8** 3 **3**] [**4** 3 **3**]

[2 12 2] [2 6 2]

[**4** 4 **6**] [**3** 3 **4**]

[**8** **3**] [**4** **3**]

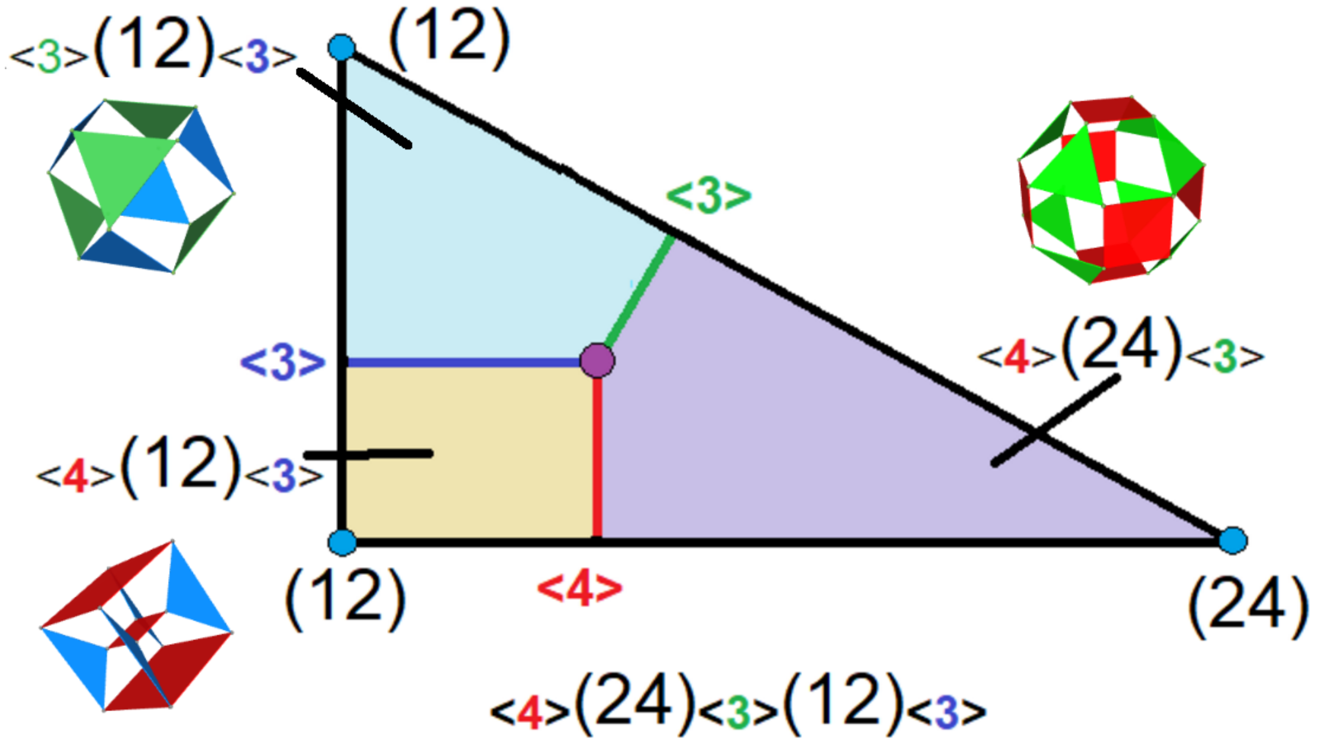
[**4** **6**] [**3** **4**]

Face-Vertex Signature ${}_{<4>}(24)_{3}(12)_{3}$

Mirror generators: {0,1,2,3}, Rotations {01,12,23}

Omnitruncation fundamental triangle

(3 edge types and 3 face types)



Apps online

Tom Ruen
hypergons.web.app

Jeff Weeks
www.geometrygames.org/KaleidoTile