Regular Polygons and Polyhedra of the Fourth Kind

MAA NCS/MinnMATYC Fall 2025 meeting, October 11, 2025

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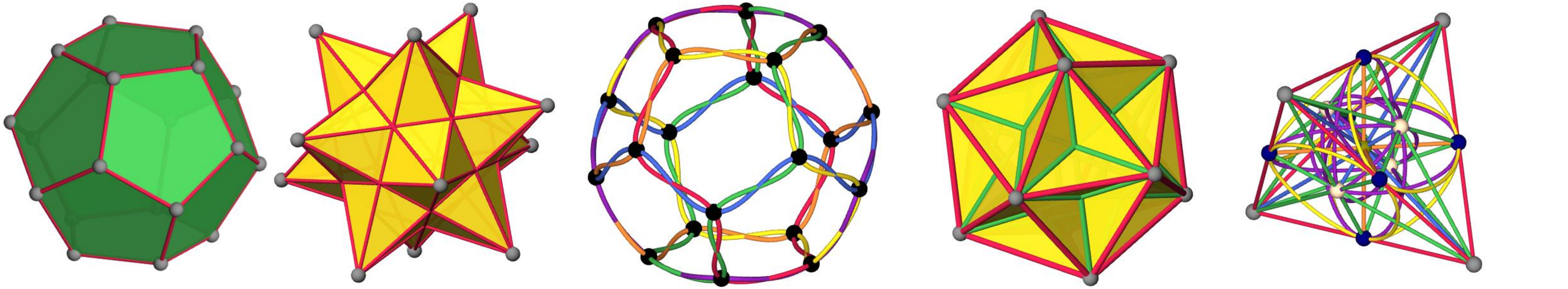
Abstract: Regular polyhedra exhibit transitivity across vertices, edges, and faces, bounded by regular faces and vertex figures. They occur in dual pairs, unless self-dual. Classical families include Platonic solids with regular convex faces (1st kind), Kepler–Poinsot polyhedra with regular star faces (2nd kind), and Petrie–Coxeter with regular skew faces (3rd kind). We propose a 4th kind, allowing faces and vertex figures to be symmetric graphs. Examples include polyhedra with faces isomorphic to K_4 (skeletal 3-simplex), K_5 (skeletal 4-simplex), $K_{2,2,2}$ (skeletal octahedron), and $2C_3$ (star of David). We survey symmetric graphs up to 15 vertices as candidate faces for future regular polyhedra.

Convex faces | **Star faces** | **Skew faces** | **Symmetric graph faces** | **Symmetric hypergraph faces**



Regular polygons and polyhedra of the 4th kind!

Regular polyhedra of *n*th kind have *n*th kind of regular faces and vertex figure.

1. **Allow regular convex faces and verfs**
   - 5 Platonic {3,p}, {p,3}, with *p*=3,4,5
2. **Allow regular star faces and verfs**
   - 4 Kepler-Poinsots {5/2,*p*}, {*p*,5/2}, *p*=3,5
3. **Allow regular skew faces or verfs**

   (a) Petrie infinite polyhedra (skew verfs): {4,6|4}, {6,4|4}, {6,6|3}

   (b) Petrial regulars (skew faces): $\{3,p\}_\pi$, $\{p,3\}_\pi$ $\{5/2,p\}_\pi$ $\{p,5/2,p\}_\pi$ with *p*=3,4,5
4. **Allow symmetric graph faces and verfs**
   - Yet unexplored polyhedra!
5. **Allow symmetric hypergraphs faces and verfs**

   (a) Regular complex polygons in C^2 (~R^4) and complex polyhedra in C^3 (~R^6).

   (b) Projective geometries PG(3,*k*) and point-line-plane configurations

First Kind: Platonic solids

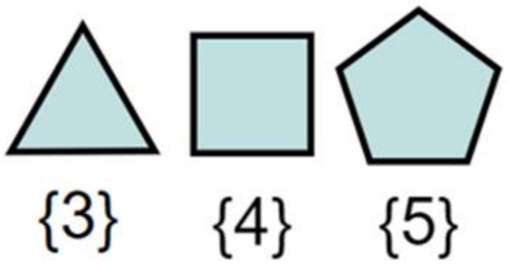
Named by Kepler after Plato (~400 BC) who associated with: *fire, earth, air, water, and aether.*

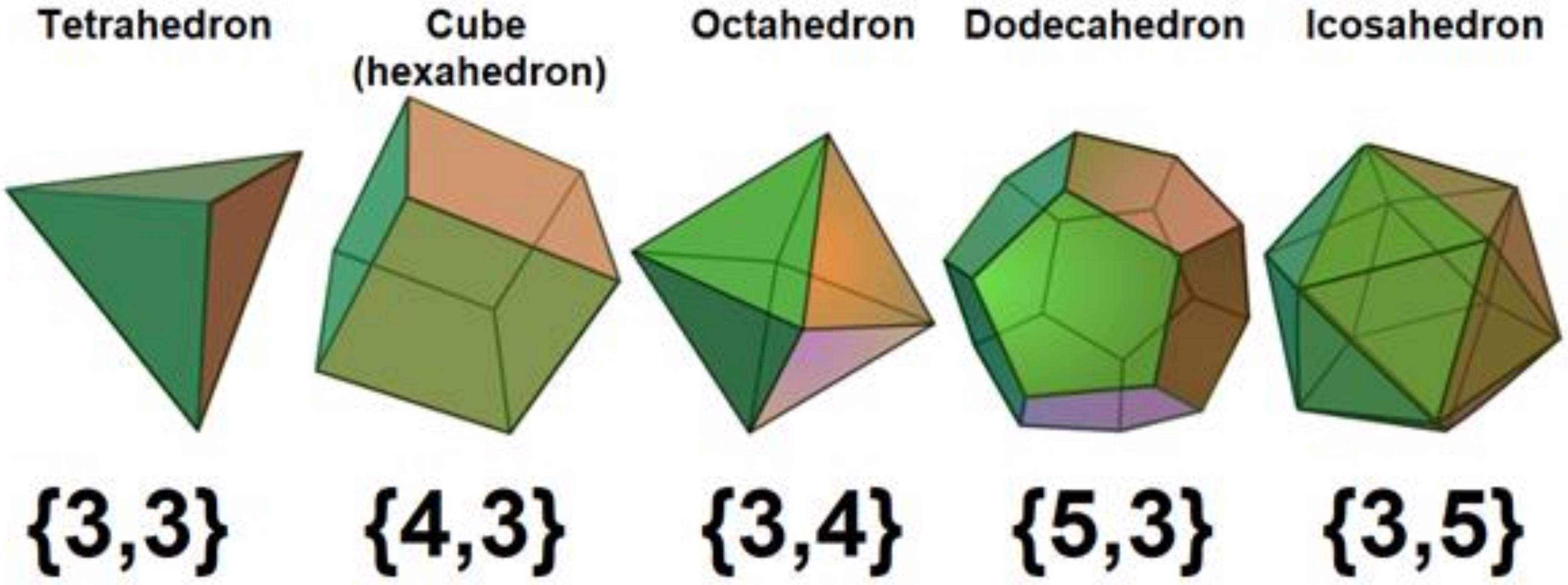
Triangles, squares and pentagons fit in a cyclic path around a vertex in 5 ways.

Schläfli symbol $\{p, q\}$ means wrap q p-gons around a vertex.

The q-gon is a "vertex figure", the cyclic path of neighboring vertices.

$\{3, 3\}$ is self-dual. $\{3, p\}$ and $\{p, 3\}$ are duals, p=4, 5.





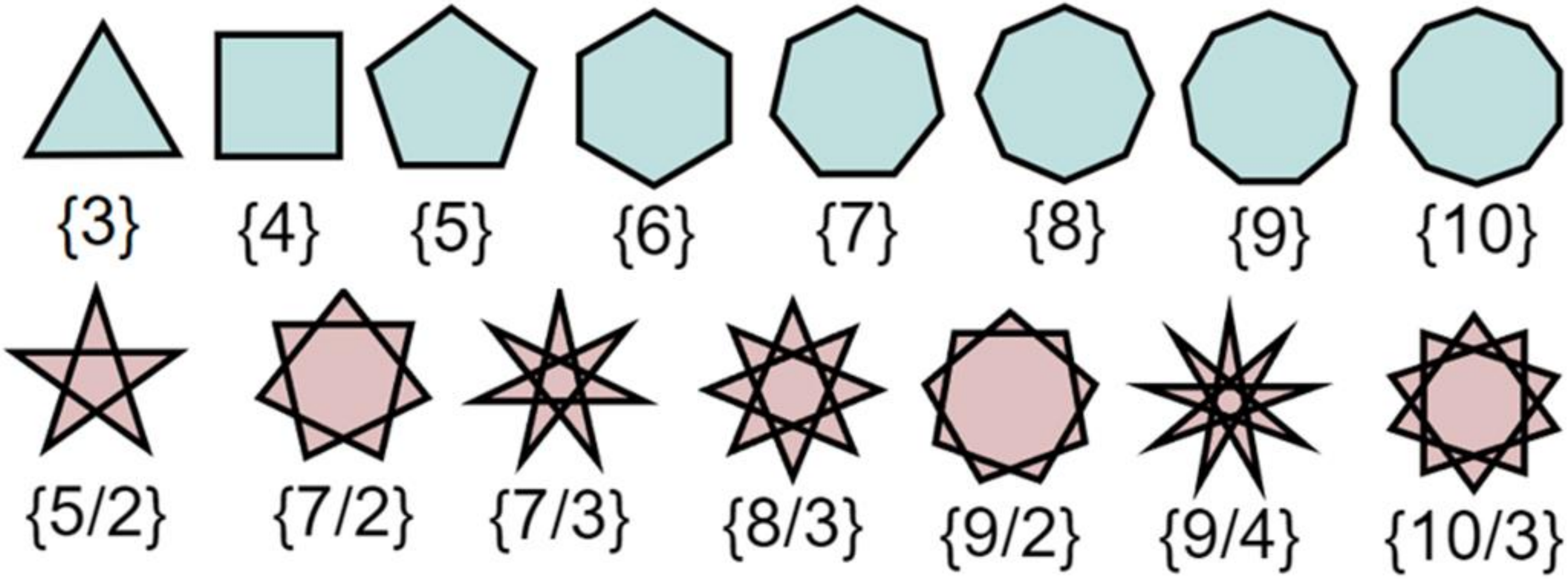
Regular polygons of the first and second kind

(First) Regular **convex** polygons, {*p*}, have *p* equal vertices and edges, p=3,4,5,…

(Second) Regular **star** polygons {*p*/*q*} have *p* vertices, edges, step *q*, gcd{*p*,*q*}=1

The *q* is also called the density, times the edge path wraps around.

Only the *pentagram*, {5/2} exists in regular polyhedra…

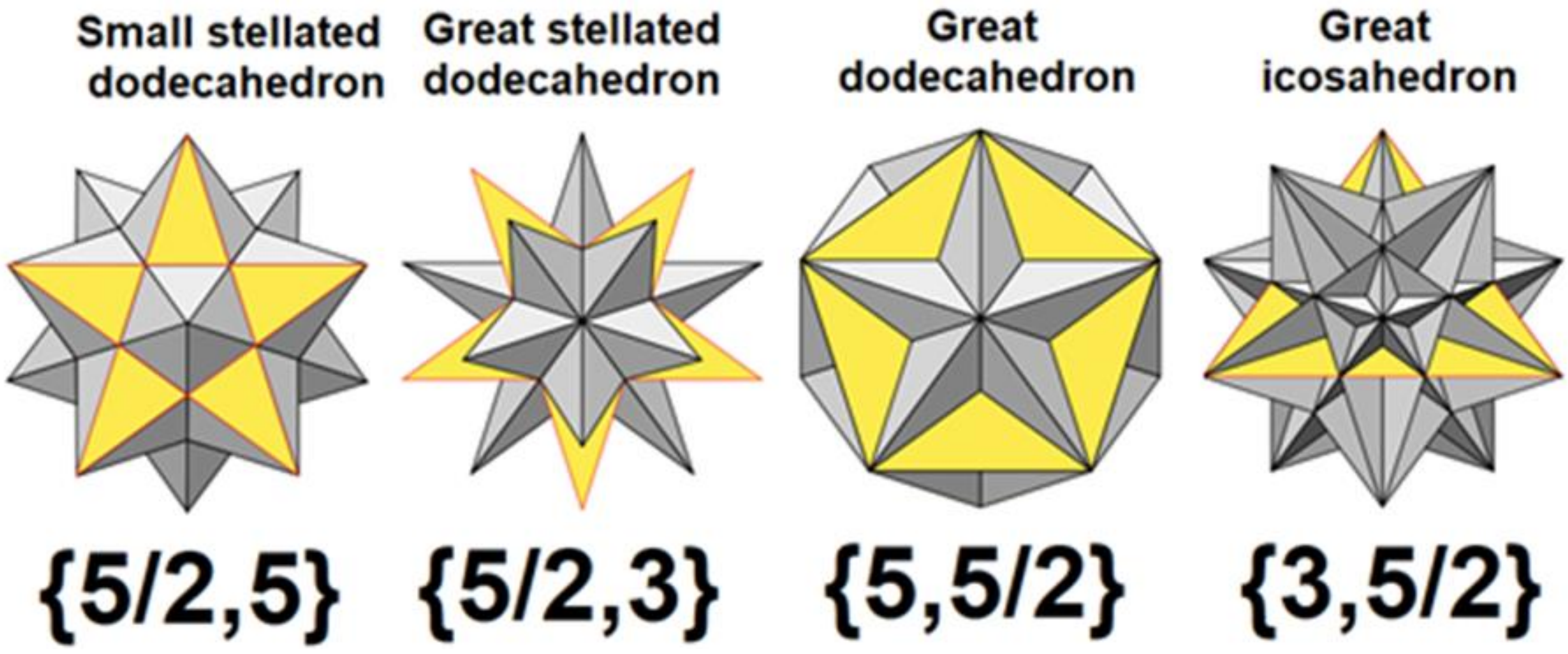


Second Kind: Kepler-Poinsot

Johannes Kepler identified in 1609, *Harmonice Mundi*

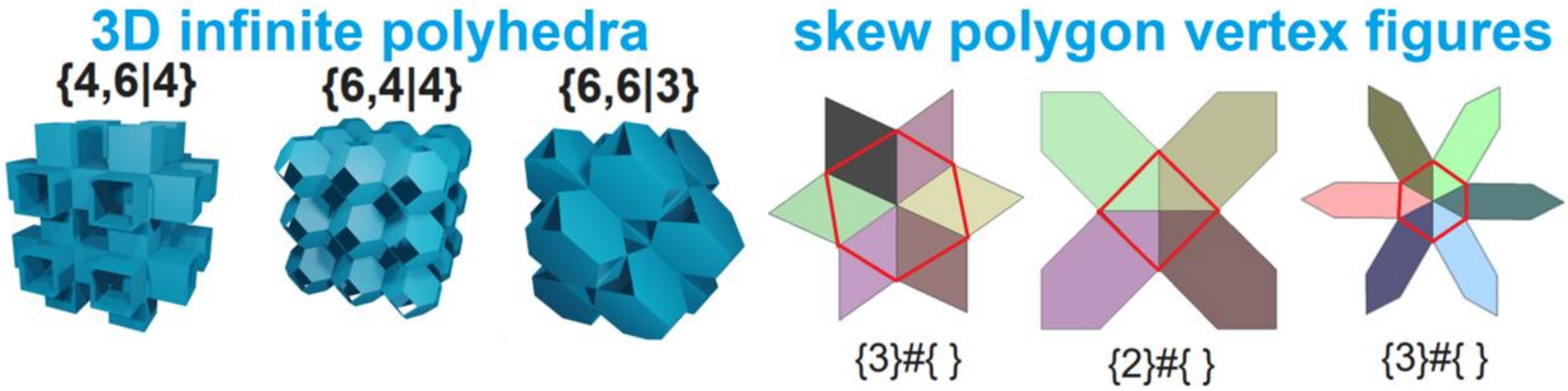
Louis Poinsot rediscovered and formalized in 1809.

One face is colored yellow for clarity



Third Kind (a): Skew vertex figures

Donald Coxeter named these infinite regular polyhedra after John Flinders Petrie who in 1926 found regular faces (squares and hexagons) fit in 3D in regular skew vertex figures.
{p, 2q| h} has 2*q* *p*-gons zig-zagging, {q}#{ }, *h*-gonal hole cycles.

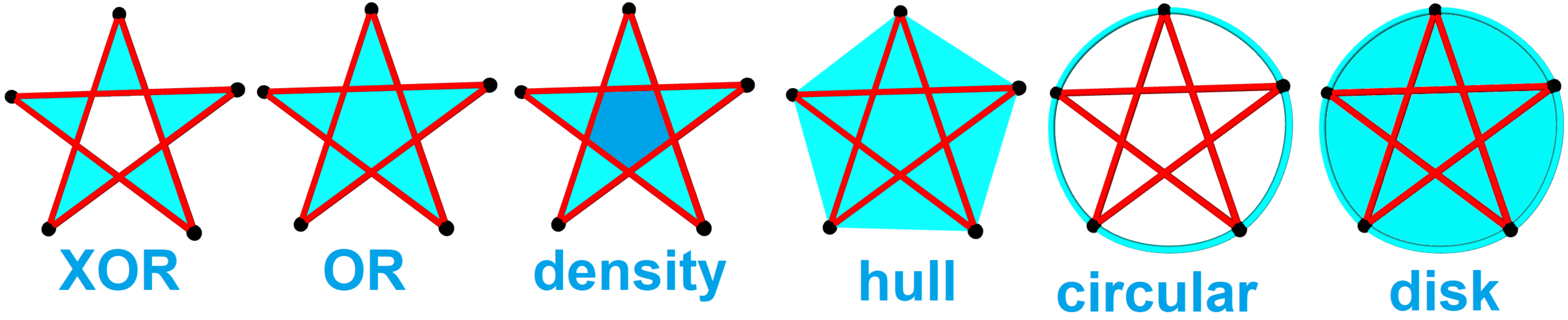


Detour: Polygon interiors

Convex polygons have clear interiors.
Star polygons are topological, not bounding space.

Representations

1. Fill by edge crossings: logical XOR, OR, density count
2. Convex hulls – polygonal interior, circular, disk interior

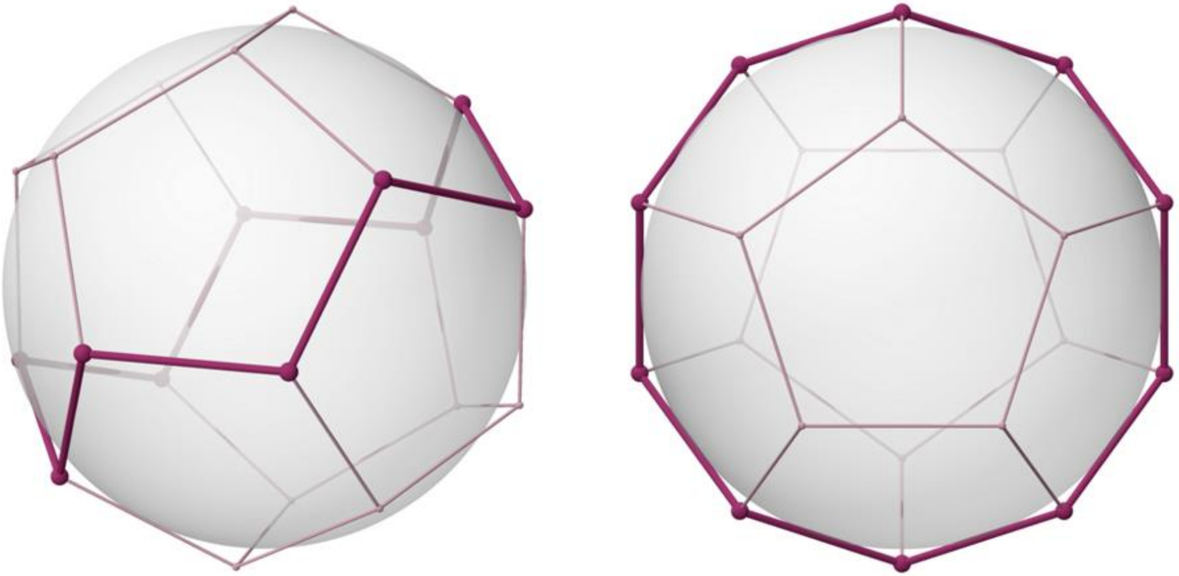


Third Kind (b): skew faces - Petrial Platonics

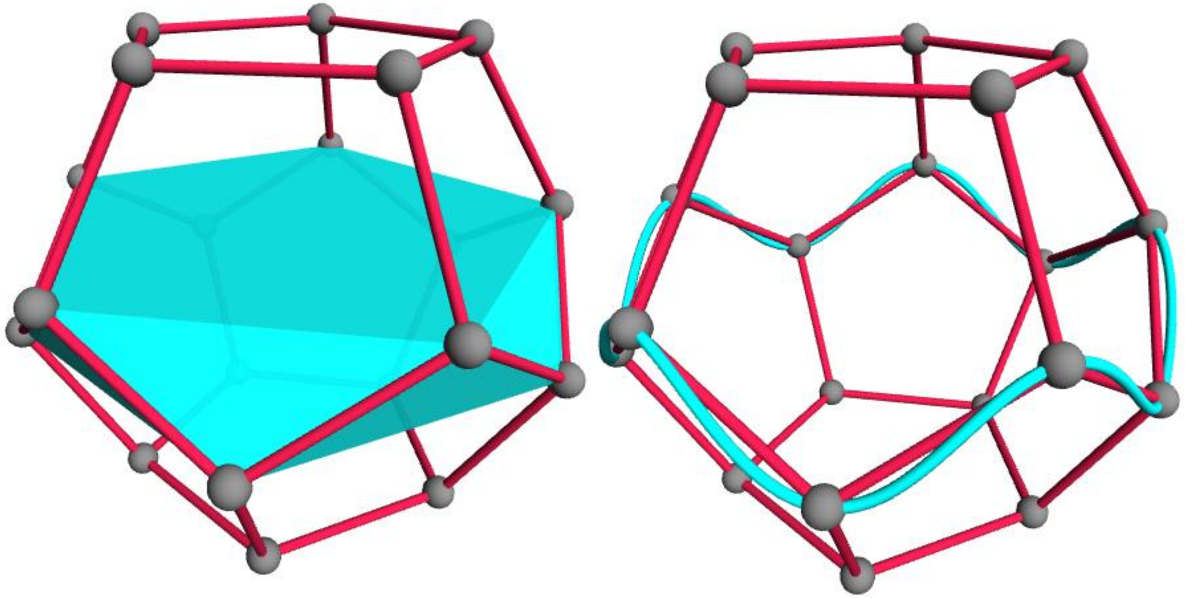
A petrial polyhedron uses skeleton of a regular polyhedron and connects faces as regular skew polygons.

Dodecahedral petrial $\{5,3\}_{\pi}$ has central skew decagon faces: {5}#{ }.

Drawing “interior” of skew decagon by convex hull or a skew circle.



Skew face: {5}#{ }



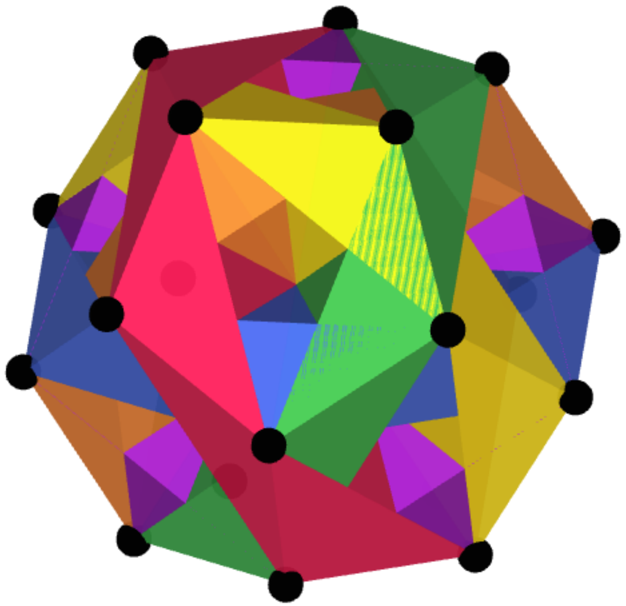
hull skew circular

Third Kind (b): skew faces

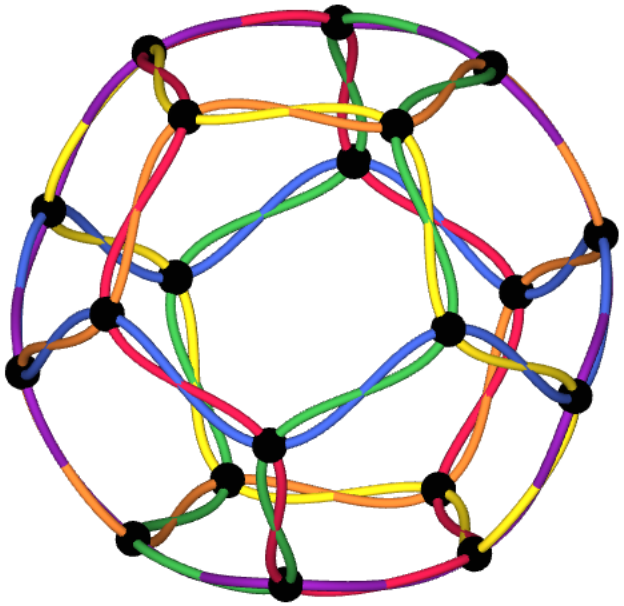
The petrial dodecahedron has the 20 vertices, and 30 edges of dodecahedron, but only 6 faces as skew decagons.

Hull render shows as interpenetrating pentagonal antiprisms

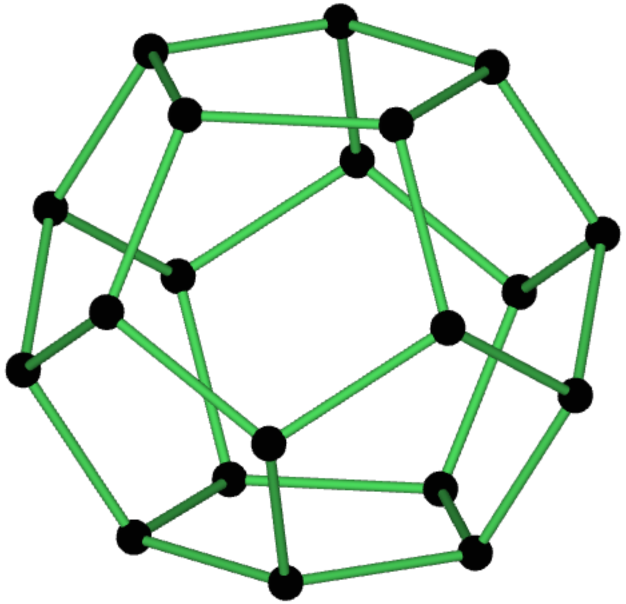
Skew circular better shows 6 colored paths, 2 faces paths per edge.



hull



skew circular

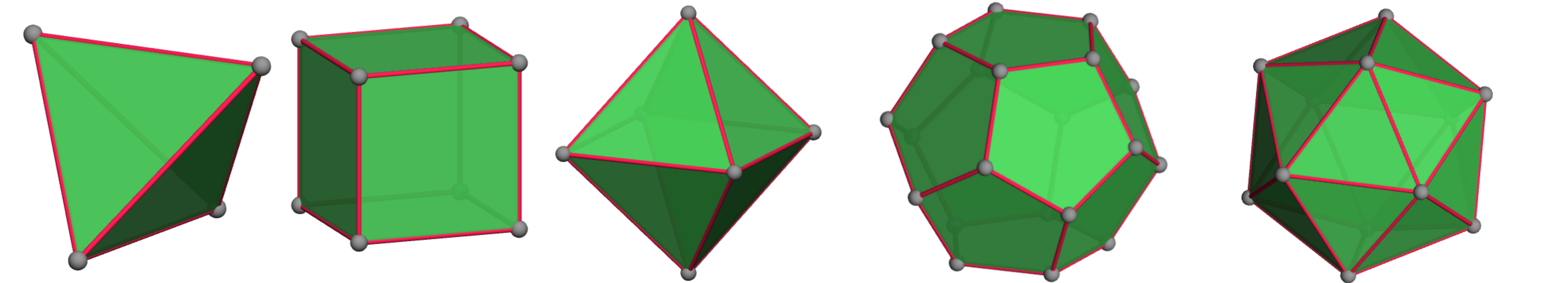


Edges only

Detour: Configuration matrix

A configuration matrix for polyhedron counts vertices, edges, and faces on the diagonal and incidences off diagonal.

Lower left counts vertices and edges of faces. Upper right on vertex figure.



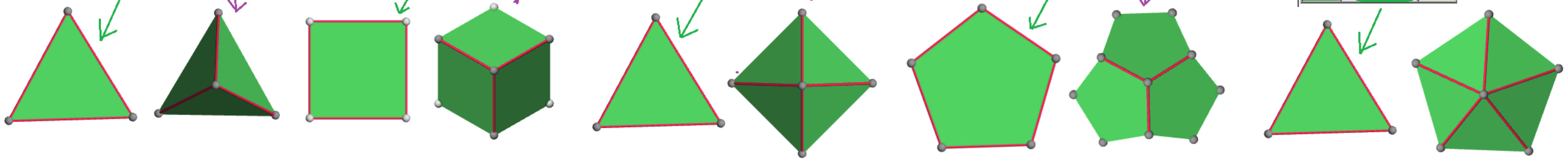
\	v1	e1	f2
v1	4	3	3
e1	2	6	2
f2	3	3	4

\	v1	e1	f2
v1	8	3	3
e1	2	12	2
f2	4	4	6

\	v1	e1	f2
v1	6	4	4
e1	2	12	2
f2	3	3	8

\	v1	e1	f2
v1	20	3	3
e1	2	30	2
f2	5	5	12

\	v1	e1	f2
v1	12	5	5
e1	2	30	2
f2	3	3	20



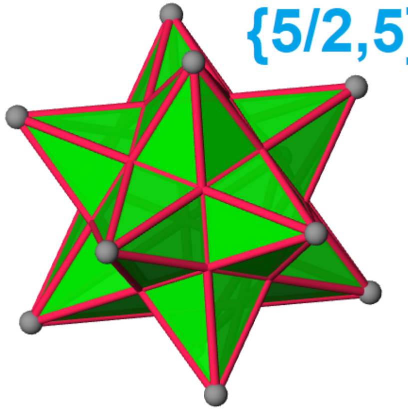
Configurations of Kepler-Poinsot

{5/2,5} and {5,5/2} have identical configuration matrices

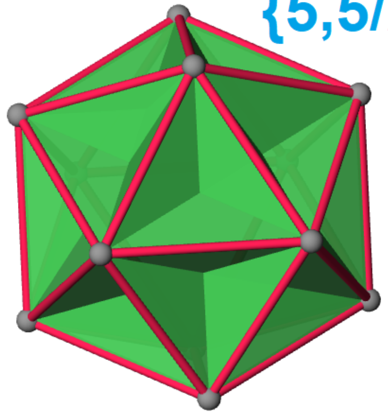
Since pentagon {5} and pentagram {5/2} same structure!

Same 12 vertices, same 12 face planes, different 30 edges.

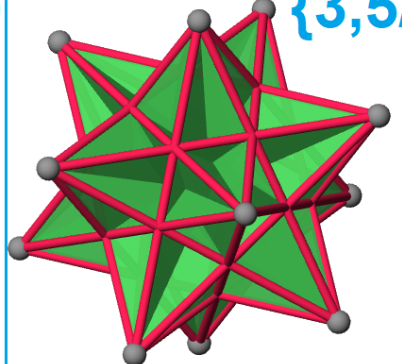
{5/2,5}



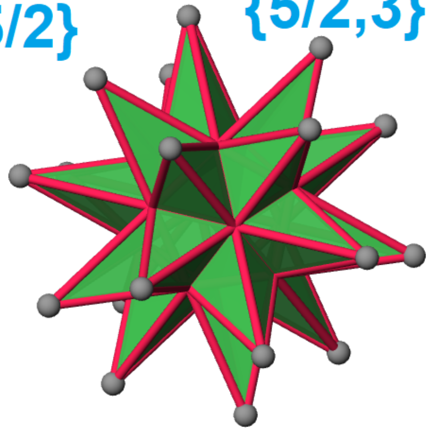
{5,5/2}



{3,5/2}



{5/2,3}



\	v1	e1	f2
v1	12	5	5
e1	2	30	2
f2	5	5	12

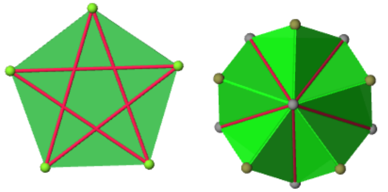
Duals Same!

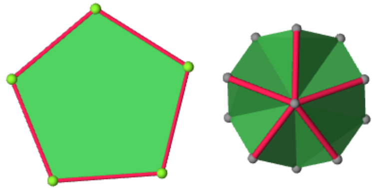
\	v1	e1	f2
v1	12	5	5
e1	2	30	2
f2	5	5	12

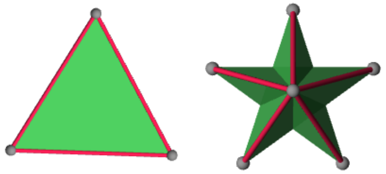
\	v1	e1	f2
v1	12	5	5
e1	2	30	2
f2	3	3	20

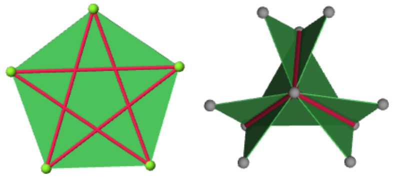
Duals

\	v1	e1	f2
v1	20	3	3
e1	2	30	2
f2	5	5	12









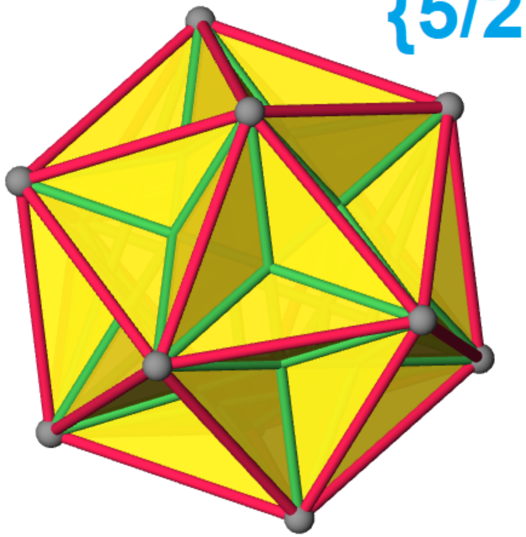
A polyhedron of 4th kind!

Union of {5/2,5} and {5,5/2} gives us $\{K_5,K_5\}$?

Each “pentagonal” face is a complete graph, K_5!

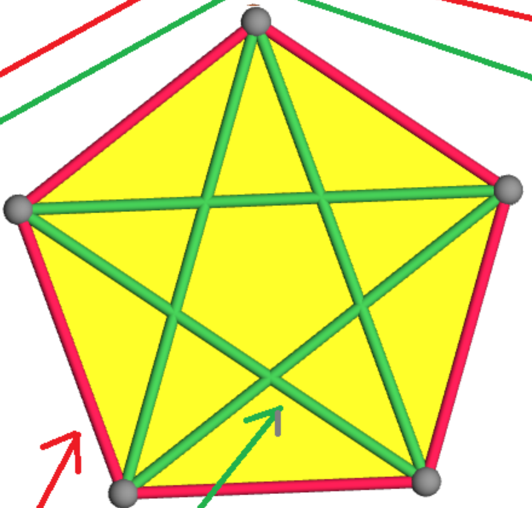
Edges (red/green) different length, but abstractly transitive, symmetry double to 240.

Full symmetry can be expressed in 6-dimension on 6-orthoplex, 4-simplex “faces”



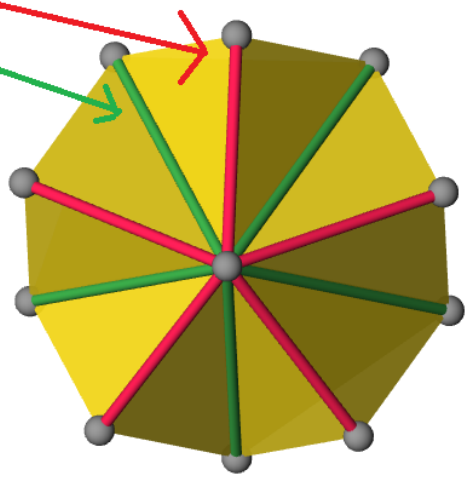
{5/2,5} U {5,5/2}

\	v1	e1	e2	f3
v1	12	5	5	5
e1	2	30	*	2
e2	2	*	30	2
f3	5	5	5	12



face

{5}U{5/2}=K5



vertex figure

dual to K5

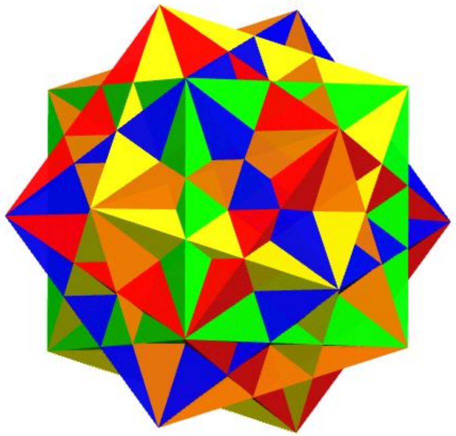
Other examples?

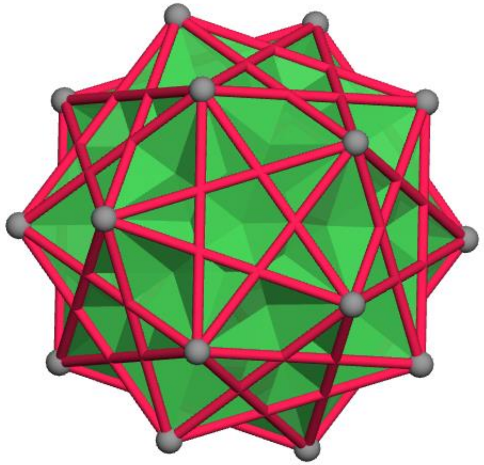
A compound of 5 cubes exist on 20 vertices of dodecahedron.

It can be interpreted as a polyhedron with square faces and a double triangle vertex figure. {4,6/2}

Its dual, compound of 5 octahedra as {6/2,4}

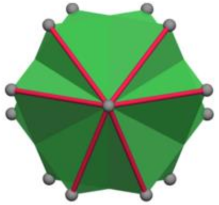
Compound 5 cubes





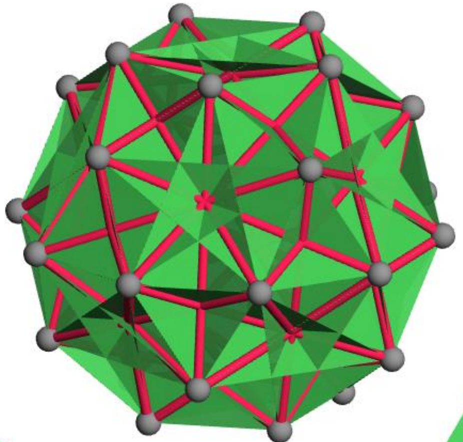
{4,6/2}

\	v1	e1	f2
v1	20	6	6
e1	2	60	2
f2	4	4	30

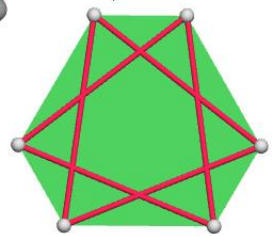


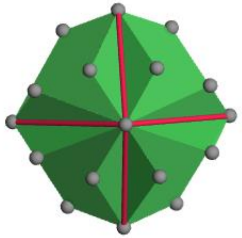
Compound 5 octahedra

{6/2,4}



\	v1	e1	f2
v1	30	4	4
e1	2	60	2
f2	6	6	20



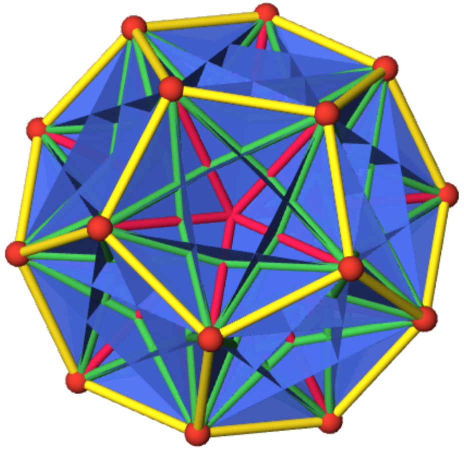


Another example

A second level faceting of a dodecahedral skeleton produces a 20 vertex, 20 face self-dual polyhedron, with 12 edges per face and 120 total edges.

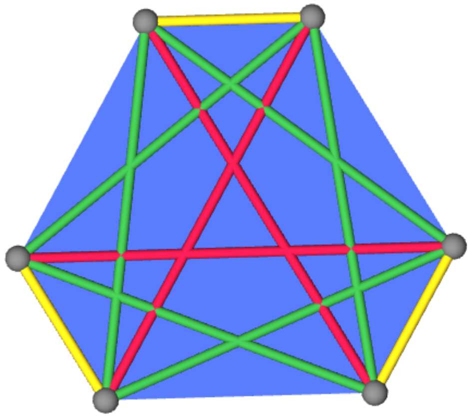
The face is isomorphic to skeleton of octahedron, graph $K_{2,2,2}$.

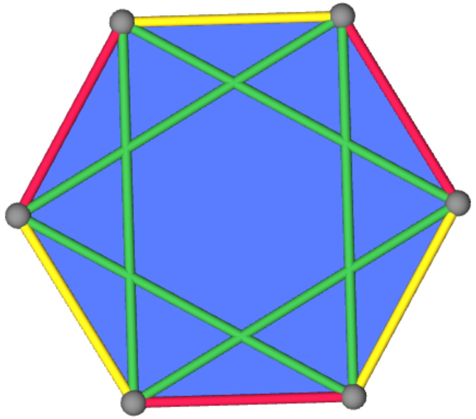
Faceting of dodecahedron



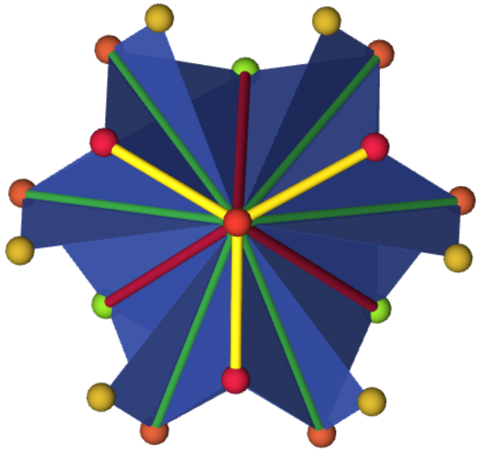
\	v1	e1	e2	e3	f4
v1	20	3	6	3	6
e1	2	30	*	*	2
e2	2	*	60	*	2
e3	2	*	*	30	2
f4	6	3	6	3	20

K2,2,2 face





Dual K2,2,2



Regular polyhedra of the 4th kind

These new polyhedra can be seen as rank 3 incidence structures composed of 1 transitivity class of vertex, edge, and face.

Each edge has 2 vertices, and incident to 2 faces.

Faces are graph: a vertices, b edges, degree $2b/a$. $(a_{2b/a}\ b_2)$

Vertex figures are graphs: c edges, d faces, degree $2d/c$. $(c_{2d/c}\ d_2)$

How many more?!
No one knows!

NEXT:

What small symmetric graphs exist?

v	c	d
2	e	2
a	b	f

v-e handshaking

vc=2e

v	c
2	e

v-f handshaking

vd=af

v	d
a	f

e-f handshaking

2e=bf

e	2
b	f

How many small symmetric graphs?

A133181 Number of distinct connected simple symmetric (edge- and vertex-transitive) graphs with n nodes

A087145 Number of distinct disconnected simple symmetric graphs on n nodes.

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founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A133181 Number of distinct connected simple symmetric (edge- and vertex-transitive) graphs with n nodes.

1, 1, 1, 2, 2, 4, 2, 5, 4, 8, 2, 11, 4, 8, 10

(list; graph; refs; listen; history; text; internal format)

OFFSET 1,4

COMMENTS
Care is needed with "symmetric" terminology, which is variously used to mean both arc-transitive and both vertex- and edge-transitive.
The symmetry means that any two vertices and any two edges are equivalent. In other words, if we have an initial labeling of the graph with vertices A and B adjacent (directly connected by an edge), we can relabel any two adjacent vertices as A and B and then relabel the remaining vertices so that new graph will be equal to the initial.
The first known difference from A286280 (connected arc-transitive graphs on n vertices) occurs at a(27), corresponding to the Doyle graph (which is both edge- and vertex-transitive but not arc-transitive). - Eric W. Weisstein, May 13 2017
By convention, empty graphs are considered edge-transitive (and hence symmetric).

LINKS
Table of n, a(n) for n=1..15.
Eric Weisstein's World of Mathematics, Arc-Transitive Graph
Eric Weisstein's World of Mathematics, Doyle Graph
Eric Weisstein's World of Mathematics, Edge-Transitive Graph
Eric Weisstein's World of Mathematics, Symmetric Graph
Eric Weisstein's World of Mathematics, Vertex-Transitive Graph

EXAMPLE
The complete graph is symmetrical.
In addition, if the number of vertices is > 3, the simple cycle through all vertices is symmetrical.
Graphs determined by vertices and edges of Platonic solids are symmetrical.
The square K X K grid with right vertices connected to corresponding left vertices and bottom vertices connected to corresponding top vertices is symmetrical.
The smallest nontrivial and non-Platonic symmetric graph is the hexagon with connected opposite vertices.
An example of a symmetrical graph with 13 vertices:

```
   0 connected to 1, 2,  3,  4
   1 connected to 0, 5,  6,  7
   2 connected to 0, 5,  8,  9
   3 connected to 0, 6, 10, 11
   4 connected to 0, 8, 10, 12
   5 connected to 1, 2, 10, 11
   6 connected to 1, 3,  8, 12
   7 connected to 1, 8,  9, 11
   8 connected to 2, 4,  6,  7
   9 connected to 2, 7, 10, 12
  10 connected to 3, 4,  5,  9
  11 connected to 3, 5,  7, 12
  12 connected to 4, 6,  9, 11
```

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(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A087145 Number of distinct disconnected simple symmetric graphs on n nodes.

0, 1, 1, 2, 1, 3, 1, 4, 2, 4, 1, 9, 1, 4, 4, 9, 1, 11, 1, 14, 4, 4, 1, 25, 3, 6, 6, 14, 1, 27,

(list; graph; refs; listen; history; text; internal format)

OFFSET 1,4

LINKS
Table of n, a(n) for n=1..31.
Eric Weisstein's World of Mathematics, Symmetric Graph

FORMULA a(n) = Sum{d|n, d<n} A133181(d). - Charlie Neder, Apr 24 2019

CROSSREFS
Cf. A133181 (number of connected symmetric graphs).
Cf. A286931 (number of not necessarily connected symmetric graphs).
Sequence in context: A338899 A362071 A194943 * A117172 A029207 A111902
Adjacent sequences: A087142 A087143 A087144 * A087146 A087147 A087148

KEYWORD nonn,more

AUTHOR Eric W. Weisstein, Aug 19 2003

EXTENSIONS
Corrected and extended by David Wasserman, Feb 14 2006
a(12)-a(31) from Charlie Neder, Apr 24 2019

STATUS approved

Vertices	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Connected	1	1	1	2	2	4	2	5	4	8	2	11	4	8	10	15	4	14	3	22	13	8	2	34	11
Unconnected	0	1	1	2	1	3	1	4	2	4	1	9	1	4	4	9	1	11	1	14	4	4	1	25	3
All	1	2	2	4	3	7	3	9	6	12	3	20	5	12	14	24	5	25	4	36	17	12	3	59	14
Cumulative	1	3	5	9	12	19	22	31	37	49	52	72	77	89	103	127	132	157	161	197	214	226	229	288	302

Dr. Marston Conder, 2021

Department of Mathematics, University of Auckland

A complete list of all connected symmetric

graphs of order 2 to 47! (previous up to 30)

Computed, indexed, but graphs are not named!

```
Below is a complete list of all connected symmetric (arc-transitive)
graphs of order 2 to 47, together with some information about their
automorphism groups. This list was constructed from the database of
all transitive groups of degree up to 47, available in Magma.

Note that for any particular order (2 to 47), the graphs are not
necessarily in the same order as in the list I created for orders
2 to 30 in April 2014.  In this one, they are ordered by valency,
and by order of the automorphism group (for each given valency).

The graphs in this list are specified by their edge-sets.
Another copy of this list is available with graphs specified by
the neighbours of each vertex.

Marston Conder
December 2021
```

```
Symmetric connected graphs of order 2


Symmetric graph 1 of order 2
Valency 1
Diameter 1
Automorphism group of order 2
Number of arcs = 2
Number of 2-arcs = 0   2-arc-transitive true
Edge-set
{ {1,2} }
```

Symmetric connected graphs with 2…15 vertices

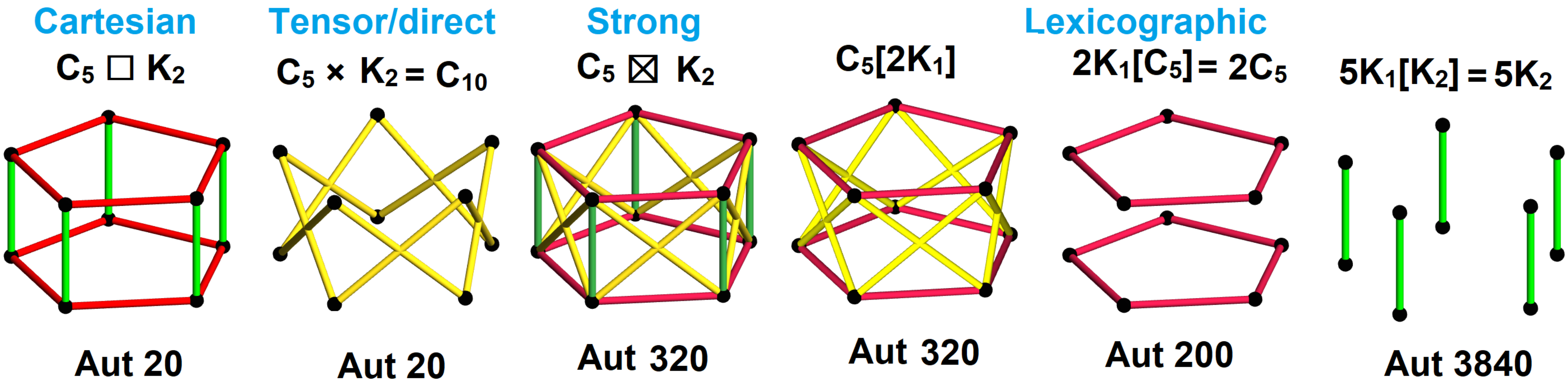
64 regular polygons of the 4^{th} kind", all but 12 are circulant. Blue odd-vertex, yellow odd-degree.

2…6	K_2 2.1	C_3 3.1		C_4 4.1	K_4 **4.2**		C_5 5.1	K_5 5.2		$C_6=K_3\times K_2$ 6.1	$K_2[3K_1]$ **6.2**	$C_3[2K_1]$ 6.3	K_6 6.4
7…9	C_7 7.1	K_7 7.2		C_8 8.1	$K_4\times K_2=Q_3$ **8.2**	$C_4[2K_1]=K_2[4K_1]$ 8.3	$K_4[2K_1]$ 8.4	K_8 8.5		C_9 9.1	$K_3\times K_3$ 9.2	$K_3[3K_1]$ 9.3	K_9 9.4
10…11	C_{10} 10.1	G(5,2) **10.2**	$K_5\times K_2$ 10.3	$C5[2K_1]$ 10.4	$K_2[5K_1]$ 10.5	$L(K_5)$ 10.6	$K_5[2K_1]$ 10.7	K_{10} 10.8		C_{11} 11.1	K_{11} 11.2		
12	C_{12} 12.1	CO 12.2	$C_4\times K_3$ 12.3	I 12.4	$K_6\times K_2$ 12.5	{12/(1,2,5)} 12.6	$C_4[3K_1]$ 12.7	$K_3[4K_1]$ 12.8	$K_4[3K_1]$ 12.9	$K_6[2K_1]$ 12.10	K_{12} 12.11		
13…14	C_{13} 13.1	{13/(1,5)} 13.2	Paley(13) 13.3	K_{13} 13.4		C_{14} 14.1	Heawood **14.2**	Co-Heawood 14.3	$C_7[2K_1]$ 14.4	$K_7\times K_2$ 14.5	$K_2[7K_1]$ 14.6	$K_7[2K_1]$ 14.7	K_{14} 14.8
15	C_{15} 15.1	$C_5\times K_3$ 15.2	$L(G(5,2))$ 15.3	Co(L(K6)) 15.4	$C5[3K_1]$ 15.5	$L(K_6)$ 15.6	$K_5\times K_3$ 15.7	$K_3[5K_1]$ 15.8	$K_5[3K_1]$ 15.9	K_{15} 15.10			

Graph products

Four important graph products: $v_{G*H} = v_G v_H$

1. **Cartesian** $G\square H$: Prism if $H=K_2$ $e_{G\square H} = v_G e_H + e_G v_H$
2. **Tensor** or direct $G\times H$: diagonal edges, $e_{G\times H} = 2e_G e_H$
3. **Strong** $G\boxtimes H$: Union of both, $e_{G\boxtimes H} = v_G e_H + e_G v_H + 2e_G e_H$
4. **Lexicographic** $G[H]$: $e_{G[H]} = v_G e_H + e_G v_H^2$ and if $H=nK_1$, $e_{G[nK1]} = n^2 e_G$



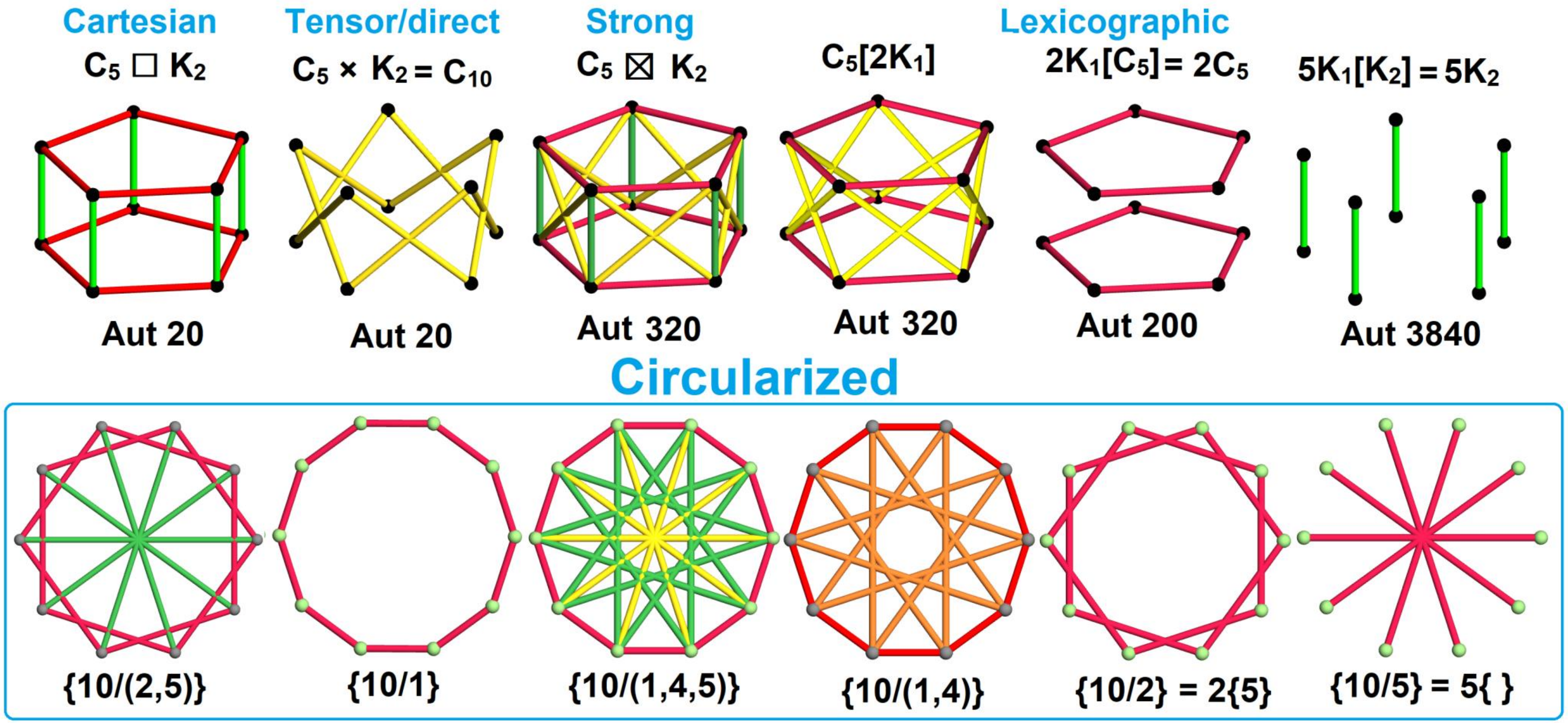
Circulant graphs $Ci_p\{q_1,q_2,\ldots,q_k\}$

Multi-star polygon $\{p/(q_1,q_2,\ldots)\} = \{p/q_1\} \cup \{p/q_2\} \ldots$ (connecting edges every q_1, $q_2 \ldots$)

Hamiltonian cyclic graphs can start with q_1=1, defining convex perimeter of p-gon.

Complete graphs index edges $\{1,2,\ldots,p/2\}$.

Compliment graphs use complete graph complement set to $\{q_1,q_2,\ldots\}$.

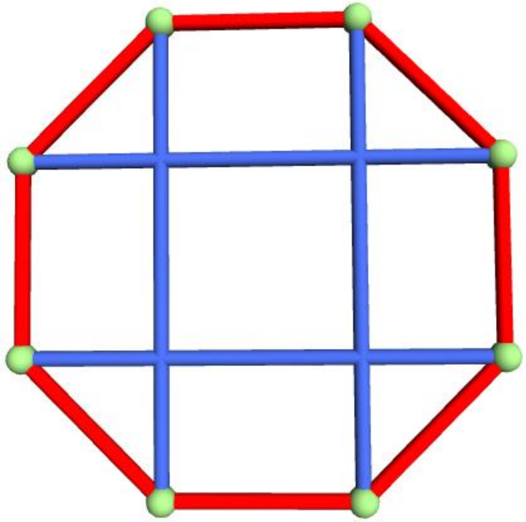


Bipartite circulant graphs

Bipartite graph circulants may include half paths of odd length, ~ symbol on length.

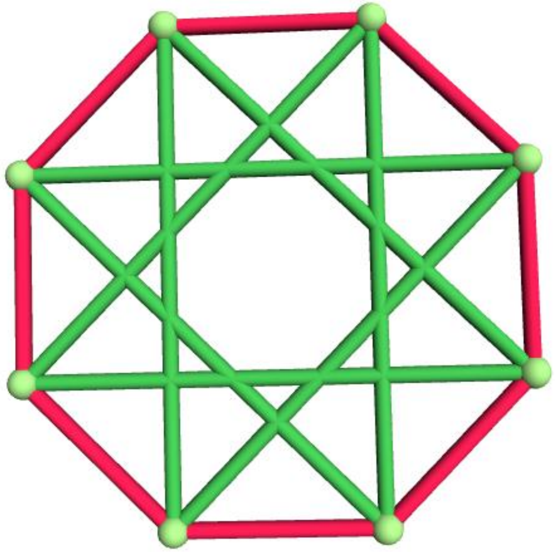
- Cubic graph Q_3 misses 4 edges of $K_{4,4}$ (cental diagonals of a cube)
- Heawood graph and its bipartite complement are both edge transitive.

$Q_3 = K_4 \times K_2$



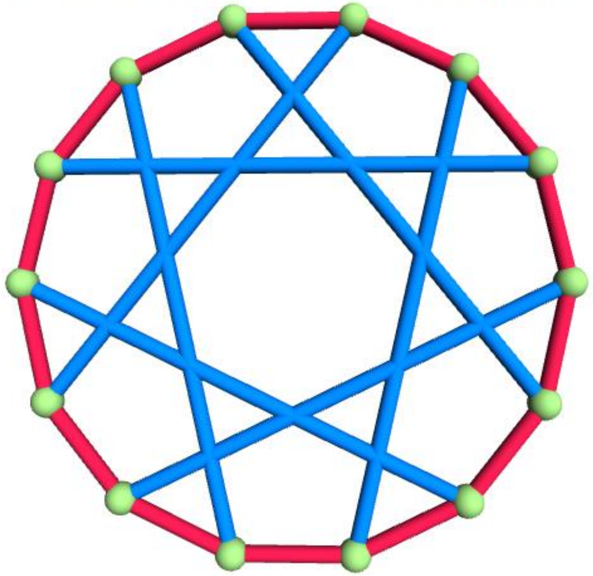
{8/(1,~3)}

$K_{4,4}$



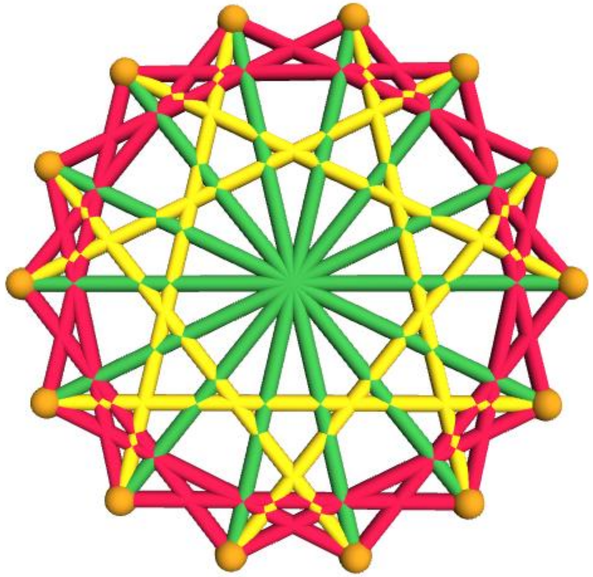
{8/(1,3)}

Heawood cubic



{14/(1,~5)}

Bipartite complement Heawood quartic



{14/(3,~5,7)}

What is a regular polygon?

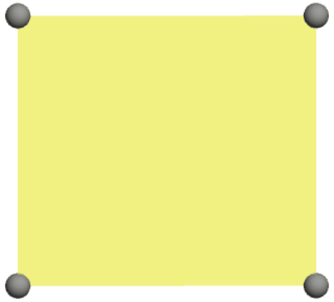
A v,e-transitive "body" attachable to other polygons edge-to-edge.

(*Full symmetry may only be displayable in higher dimensions*)

(1-polytope)

Edgeless "tetron"

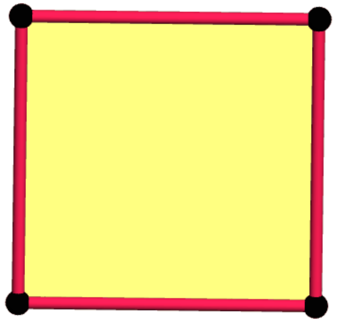
{4/0}



First Kind

Flat square

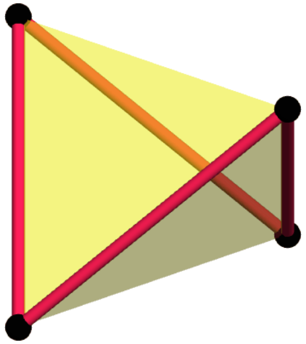
{4/1}



Third kind

Skew "square"

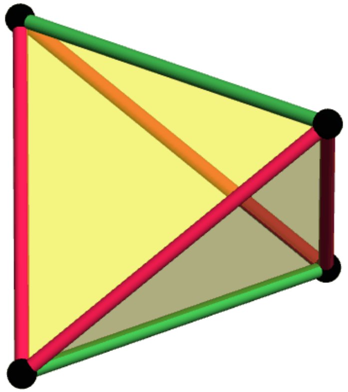
{2}#{ }



Fourth kinds

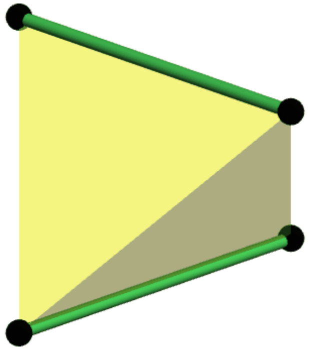
Tetrahedral "square"

{4/(1,2)}



Compound "square"

{4/2}



Regular polygons and polyhedra of the 4^{th} kind!

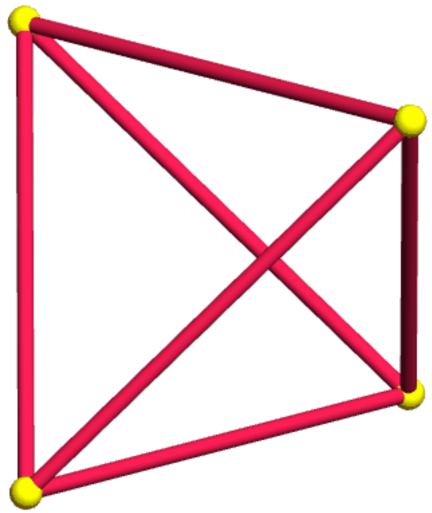
- Regular polyhedra of *n*th kind have *n*th kind of regular faces and vertex figure.

1. **Allow regular convex faces and verfs**
   - 5 Platonic {3,p}, {p,3}, with p=3,4,5
2. **Allow regular star faces and verfs**
   - 4 Kepler-Poinsots $\{5/2,p\}$, $\{p,5/2\}$, p=3,5
3. **Allow regular skew faces or verfs**
   - (a) Petrie infinite polyhedra (skew verfs): {4,6|4}, {6,4|4}, {6,6|3}
   - (b) Petrial regulars (skew faces): $\{3,p\}_\pi$, $\{p,3\}_\pi$ $\{5/2,p\}_\pi$ $\{p,5/2,p\}_\pi$ with p=3,4,5
4. **Allow symmetric graph faces and verfs**
   - Finally begun being explored!
5. **Allow symmetric hypergraphs faces and verfs**
   - (a) Regular complex polygons in C^2 (~R^4) and complex polyhedra in C^3 (~R^6).
   - (b) Projective geometries PG(3,k) and point-line-plane configurations

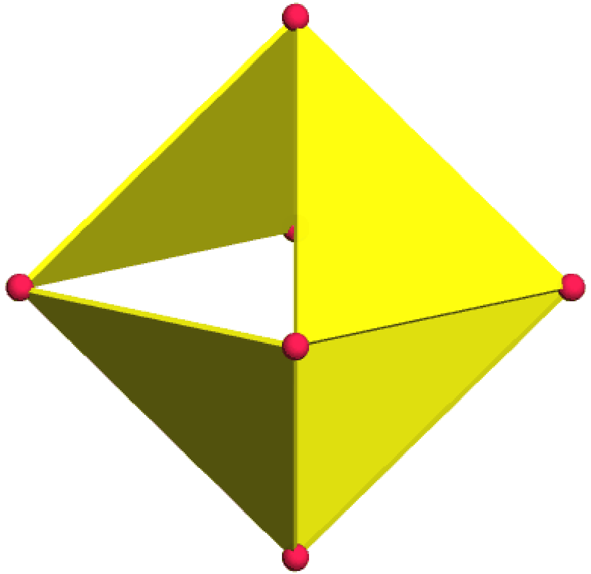
Regular Polygons of the 5th kind

Regular polygons as symmetric hypergraphs!

Complete Quadrilateral	Complete Quadrangle
$(4_3\ 6_2)$	**$(6_2\ 4_3)$**



\	v1	e1
v1	4	3
e1	2	6



\	v1	e1
v1	6	2
e1	3	4

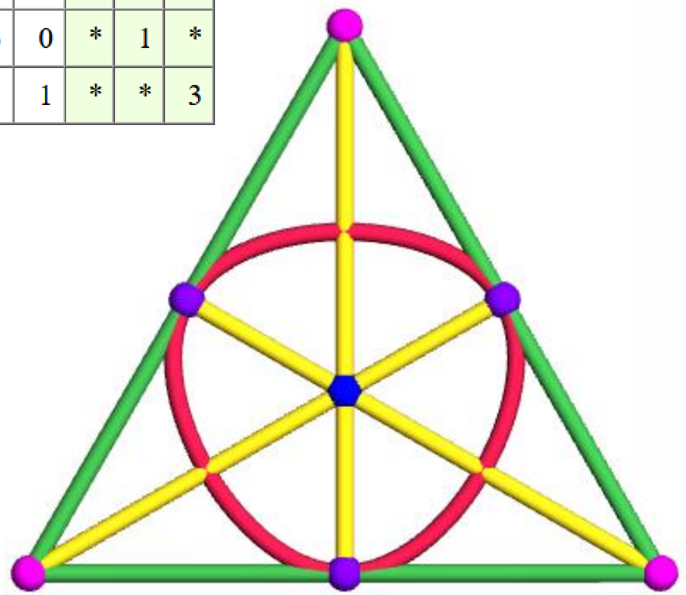
Projective Geometries

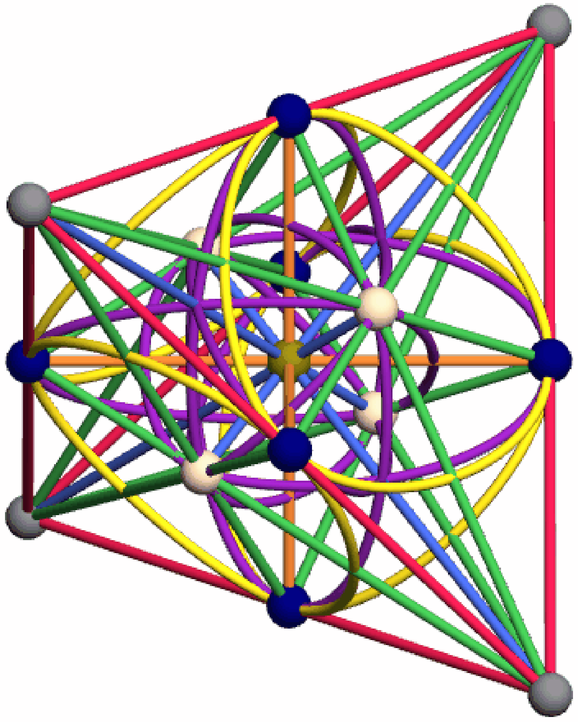
PG(2,2) is a Fano Plane, configuration (7_3), 7 vertices, 7 lines. (See PG incidence structures)

PG(3,2) is "polyhedron" with 15 vertices, 35 PG(1,2) trionic edges, and 15 PG(2,2) faces.

Both self-dual they can be drawn in triangle and tetrahedron with 3-vertex lines and circles.

\	v1	v2	v3	e1	e2	e3
v1	3	*	*	2	0	1
v2	*	3	*	1	1	1
v3	*	*	1	0	0	3
e1	2	1	0	3	*	*
e2	0	3	0	*	1	*
e3	1	1	1	*	*	3





\	v1	v2	v3	v4	e1	e2	e3	e4	e5	e6	f7	f8	f9	f10
v1	4	*	*	*	3	1	0	3	0	0	3	1	3	0
v2	*	6	*	*	2	0	1	1	1	2	2	2	2	1
v3	*	*	4	*	3	1	3	0	0	0	1	3	3	0
v4	*	*	*	1	0	4	0	0	3	0	0	0	6	1
e1	1	1	1	0	12	*	*	*	*	*	1	1	1	0
e2	1	0	1	1	*	4	*	*	*	*	0	0	3	0
e3	0	1	2	0	*	*	6	*	*	*	0	2	1	0
e4	2	1	0	0	*	*	*	6	*	*	2	0	1	0
e5	0	2	0	1	*	*	*	*	3	*	0	0	2	1
e6	0	3	0	0	*	*	*	*	*	4	1	1	0	1
f7	3	3	1	0	3	0	0	3	0	1	4	*	*	*
f8	1	3	3	0	3	0	3	0	0	1	*	4	*	*
f9	2	2	2	1	2	2	1	1	1	0	*	*	6	*
f10	0	6	0	1	0	0	0	0	3	4	*	*	*	1